Australasian Journal of Philosophy
2009; pp. 1–20, i*First* article

Routledge
Taylor & Francis Group

CREATIVITY, FREEDOM, AND AUTHORITY: A NEW PERSPECTIVE ON THE METAPHYSICS OF MATHEMATICS

Julian C. Cole

I discuss a puzzle that shows there is a need to develop a new metaphysical interpretation of mathematical theories, because all well-known interpretations conflict with important aspects of mathematical activities. The new interpretation, I argue, must authenticate the ontological commitments of mathematical theories without curtailing mathematicians' freedom and authority to *creatively* introduce mathematical ontology during mathematical problem-solving. Further, I argue that these two constraints are best met by a metaphysical interpretation of mathematics that takes mathematical entities to be constitutively constructed by human activity in a manner similar to the constitutive construction of the US Supreme Court by certain legal and political activities. Finally, I outline some of the philosophical merits of metaphysical interpretations of mathematical theories of this type.

1. A Puzzle

Reflecting on my experiences as a research mathematician, three things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and referred to mathematical entities.[1] Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over *any* mathematical entities I wished, provided it served a legitimate mathematical purpose. And third, the authority I felt I had to engage in both types of activities. Most mathematicians will recognize these features of their everyday mathematical lives.

Observations like my first are made frequently. They are routinely among the evidence that is cited in favour of realist or Platonist interpretations of mathematical theories. That is, they are cited as a reason to accept that mathematical objects—or domains or structures—(a) exist, (b) are abstract entities and, in the case of mathematical domains, are made up of abstract

[1]This is not to say that it was a trivial matter to engage in such activities, but rather that I did not experience any of the intellectual discomfort that should have accompanied them if some realist, Platonist, fictionalist, or modal nominalist interpretation of mathematical theories were correct.

Australasian Journal of Philosophy
ISSN 0004-8402 print/ISSN 1471-6828 online © 2009 Australasian Association of Philosophy
http://www.tandf.co.uk/journals DOI: 10.1080/00048400802598629

entities, and (c) are independent of rational activities[2]—both mental and physical; that is, they would exist and have the mathematical properties that they do even if the actual world contained no such activities. Strictly speaking, observations like my first motivate the acceptance only of (a)—*the existence thesis*. Yet, because of widely recognized features of mathematical theories, acceptance of the existence thesis is nearly always accompanied by the acceptance of (b)—*the abstractness thesis*—and an often implicit acceptance of (c)—*the independence thesis*. Surprisingly, observations like my second and third are made less frequently than those like my first.[3] So, let me say a little more about those.

Mathematicians are intellectual problem solvers. Mathematical problems frequently require creative solutions. Sometimes it is necessary for a mathematician to introduce a mathematical theory whose variables range over a previously uncharacterized collection of mathematical entities in order to solve the problem on which he or she is working. An interesting example of this is William Hamilton's introduction of quaternions as part of his solution to the problem of how to multiply ordered 3-tuples. His move of postulating entities with a non-commutative algebra was both bold and brilliantly creative. It is a perfect example of how new collections of mathematical entities are introduced by mathematicians in the course of creative, intellectual problem-solving.[4]

For me, and presumably most mathematicians, the process of solving a problem with the assistance of new mathematical entities involves two components: *discovery* and *creation*. It is by a process of discovery that you determine what features the new entities must have in order to serve their intended purpose. During this process, you freely postulate—or attempt to postulate[5]—candidate entities and explore their suitability for the task at hand. Sometimes, appropriate candidates occur to you almost immediately. On other occasions, you must attempt numerous postulations before you introduce appropriate entities. The features you provide—or seek to provide—each newly postulated collection of entities are informed by what has been discovered during previous attempts. Despite this, each successful attempt is a relatively free, creative act of postulation. To return to Hamilton's introduction of quaternions, it is well documented that his early efforts to solve the problem of multiplying ordered 3-tuples centred around

[2]For convenience, I shall simply talk about rational activities rather than rationally constrained activities. I have in mind a broadly pragmatic notion of rationality, one which places very few constraints on rational activities other than the basic constraints of logic.

[3]I find this surprising, because such observations could be used as a motivation for mathematical fictionalism.

[4]Hamilton's description of how he came to introduce quaternions can be found in Hamilton [1844].

[5]Attempts to postulate mathematical entities fail when the features with which you seek to endow the entities force incoherence into any characterization of them. I have in mind here the notion of *coherence* that governs contemporary classical mathematics, the tradition within which I work. Deductive consistency and set-theoretic satisfiability are good models for this notion of coherence. Where these two notions come apart, set-theoretic satisfiability is the better model. A formal treatment of coherence can be found in Shapiro [1997]. There are certainly contemporary mathematicians, working in non-classical traditions, whose theory construction is not governed by this classical notion, and who typically work with stronger or weaker notions of coherence. Even the most ardent dialetheist must work with some such constraint, however, for to do otherwise would be to accept trivial theories—theories from which everything follows. These can serve no useful purpose. In this paper, I shall continue to work with the classical notion of coherence. Parallel claims to those made in this paper could be used to provide a metaphysical account of non-classical—including dialetheic and intuitionist—mathematics.

attempts to postulate 'three-dimensional' extensions of the complex numbers. It was only after the failure of these attempts—for all such attempts are fated to result in incoherence[6]—that he thought to postulate a 'four-dimensional' extension of the complex numbers. Considering my own experiences, I can imagine that when Hamilton finally postulated quaternions, it literally felt to him as though he was creating quaternions. Those familiar with this creative phenomenology will understand why Richard Dedekind was predisposed to describe numbers as 'free creations of the human mind' [1888: 21].

So, mathematical postulation involves both discovery and creation. Yet this discovery is not of, or at least doesn't feel like it is of, a Platonic realm. If the postulated entities were to exist in a Platonic realm, this discovery would be an unnerving *happenstance* from the phenomenological perspective of the postulating mathematician. Rather, mathematical postulation is guided by the problem to be solved and a need for coherence. Any suggestion that such guidance is the guidance of a Platonic realm would be misguided. In the course of offering various epistemological challenges to Platonism, a variety of authors have given good reasons to reject the idea that a Platonic realm could guide mathematicians in this kind of way.

This having been said, it should be recognized that, once a new collection of mathematical entities has been postulated, it feels as though they have objective features that you must endeavour to determine.[7] Among the things that you must determine is whether *certain* positive existential statements concerning the postulated entities are true. Accordingly, there is a change in your phenomenology from creation informed by discovery to that of pure discovery. Actually, the creative aspects of mathematical theorizing are quite minimal. Yet, in providing an interpretation of mathematical theories, they should not be ignored.

The creative phenomenology of mathematical postulation is all the more forceful for the *freedom*, i.e., lack of constraint, that is felt to govern it. It feels as though there are only two types of constraints on such postulation. The most important constraints are those imposed by the problem to be solved. Creative postulation is also guided by a need to generate a coherent theory. Generally speaking, at least nowadays (history shows that things have not always been this straightforward), newly postulated mathematical entities are accepted by mathematicians provided there is a coherent way of characterizing them and they serve some legitimate mathematical purpose.

While I was solely a mathematician, the thought that there might be something illegitimate about any of these creative or ontologically committal activities was alien to me. Referring to and making existential assertions concerning both previously characterized and newly postulated mathematical entities were everyday, unproblematic activities. To question these activities would have been to question an authority that every mathematician implicitly recognizes: an *authority* to engage in *these very*

[6]The previous footnote contains a discussion of coherence. Hurwitz's theorem [1898] tells us that it is only possible to coherently characterize normed division algebras on the reals with 1, 2, 4, and 8 'dimensions'.
[7]At this stage, Gödel's metaphor of mathematical truths 'forcing themselves upon you' is to some extent appropriate.

activities constrained only by coherence and the problem to be solved. A lack of recognition of the legitimacy of these activities could only signal a misunderstanding of the very nature of mathematical theorizing, for the creative freedom and authority mentioned are fundamental to that activity. From the perspective of a working mathematician, such activities do not stand in need of an external ground. No wonder, then, that mathematicians do not respond favourably to the inquiries of philosophers with respect to these aspects of their activities.

Qua metaphysician, however, I can fully understand why so many philosophers *do* question the legitimacy of these activities and the authority that mathematicians enjoy to engage in them constrained only by coherence and the problem to be solved. Primarily, this is because philosophers lack a metaphysical interpretation of mathematical theories that comfortably authenticates the observations with which I began. The prevalence of existential pure mathematical statements and variables ranging over mathematical entities in mathematical theories is most comfortably interpreted by philosophers as evidence for some variety of realism or Platonism. Yet, such metaphysical interpretations do not authenticate the creative aspects of mathematical theorizing. Additionally, to offer a traditional realist or Platonist interpretation of mathematical theories is, at least *prima facie*, to curtail mathematicians' freedom and authority.[8] In virtue of their limited size and independence from mathematical—and all other rational—activities, the mathematical realms countenanced by such interpretations place further constraints on mathematical theories than the two discussed above. In particular, mathematical theories are responsible for accurately representing these independent mathematical realms. And mathematicians' authority—particularly to postulate new mathematical entities—is legitimate only if they are epistemologically in tune with these realms. The worries many have on this front are well-known (see Benacerraf [1973] and Field [1989]).[9]

On the other hand, to offer an error-theoretic or fictionalist interpretation of mathematical theories—the only well-known class of interpretations that fully authenticates mathematicians' creative freedom and authority—is to downplay my first observation. Proponents of such interpretations insist that we should not take seriously the ontological commitments of mathematical theories; they insist that such theories should be interpreted in a non-literal way (e.g., fictionally, metaphorically, figuratively, or as make-believe). The evidence does not support the hypothesis that we use the relevant statements non-literally.[10]

[8]*Traditional realisms* or *Platonisms* are those that *do not* countenance extremely large mathematical realms. For examples of *non-traditional* Platonisms, see Balaguer [1998] and Shapiro [1997].

[9]The considerations I offer against endorsing a non-traditional realism or Platonism are less than ideal. A different argument is needed to adequately support the thesis that some such realism or Platonism is not an optimal metaphysical interpretation of mathematical theories. This argument involves appreciating that once one acknowledges the type of mathematical realm countenanced by my favoured interpretation, the mathematical realms countenanced by non-traditional realisms and Platonisms are superfluous metaphysical extravagancies. See Cole [2008].

[10]John Burgess has done the most to establish this thesis (see, e.g., Burgess [2004] and Burgess and Rosen [2005]). A detailed defence would take us well beyond the scope of this paper.

So, philosophers of mathematics—who typically endorse realist, Platonist, fictionalist, or modal nominalist interpretations—face the following troubling puzzle:

> The *freedom* and *authority* that mathematicians feel they enjoy to *creatively* postulate mathematical entities do not accord well with realist or Platonist interpretations of mathematical theories. Yet, the intellectual ease with which mathematicians ontologically commit themselves to mathematical entities does not accord well with fictionalist or modal nominalist interpretations of mathematical theories.

How should we respond?

2. Towards a Solution to the Puzzle

We need a new metaphysical interpretation of mathematical theories, one that authenticates the *prima facie* ontological commitments of those theories without curtailing mathematicians' creative freedom and authority. Such an interpretation must, therefore, accept the existence thesis, i.e., it must accept the existence of mathematical entities. So, to overcome Platonism's drawbacks, it must either reject the abstractness thesis or the independence thesis.

The heart of the abstractness thesis is the thought that mathematical entities are not constituents of the spatio-temporal world, which fits well with our everyday experience of that world. For example, we do not find the number two—as opposed to pairs of items—anywhere in that world. More importantly, the abstractness thesis is guaranteed true by the immensity of mathematicians' ontological commitments. While one *might* plausibly argue that there are 2 to the continuum-many spatio-temporal entities,[11] there is no chance that the cardinality of spatio-temporal entities is an inaccessible cardinal. Yet set-theorists routinely and legitimately affirm that sets with inaccessible cardinality exist.

So, we are left with the option of rejecting the independence thesis, i.e., rejecting the independence of mathematical entities from rational activities. Furthermore, it is plausible that this option can be of assistance in solving our puzzle. If mathematical entities depend on mathematical activities, then, in some sense, mathematicians do create mathematical entities. Further, the extra—independent representational—constraints that would accompany a traditional realist or Platonist interpretation of mathematical theories do not constrain those theories. Additionally, the freedom of mathematicians to use variables that range over any collection of mathematical entities that can be coherently characterized is secured. All that remains to be secured is their authority to do so. So, I suggest that in looking for a solution to our puzzle, we explore the possibility that mathematical entities are dependent

[11]Consider the mereological fusions of continuum-many spatio-temporal atoms.

on mathematical activities; that is, they would not have existed if the actual world had not contained mathematical activities.

In order to turn this suggestion into a determinate philosophy of mathematics, it is necessary to explore the senses in which entities can depend on rational activities so that we can identify the appropriate type of dependence for mathematical entities. Consequently, in §3, I shall explore some senses in which entities can depend on rational activities. In §4, I shall use the discussion in §3 to motivate a philosophy of mathematics that I believe offers the optimal solution to our puzzle. Finally, in §5, I shall briefly explicate some of the details of this philosophy of mathematics and discuss some of its philosophical merits.

3. Social Construction and Dependence

To my knowledge, there are two distinct ways in which existent entities can depend on rational activities. Consider first such items as cars, scissors, and telephones. These are spatio-temporal entities that have, typically, been manufactured to serve some specific functional purpose. Of course, it is possible for items that occupy the functional roles of these kinds of items to exist without rational beings having taken a role in bringing them into existence. Indeed, where the functional role in question is simple enough, some such items are likely to exist (e.g., tree stump chairs). Yet we can simply ignore such instances and call the remaining such items *artefacts*. No one should doubt that many artefacts exist[12]—the vast majority of items that occupy these types of functional roles are artefacts. Nor should anyone doubt that artefacts would not exist in this world if there were no rational activities in it. Thus, artefacts are existent entities that depend on rational activities.

Further, the primary mechanism of artefacts' dependence on rational activities is well understood. Artefacts are *causally dependent* on rational activities in that rational activities causally manipulate the spatio-temporal world to bring artefacts into existence. Sally Haslanger [1995: 98] defines a *causal social construct* to be an item for which social factors play a causal role in bringing it into existence or, to some substantial extent, in its being the way that it is. Artefacts are causal social constructs in Haslanger's sense.

Next, consider the US Supreme Court, the Microsoft Corporation, the Royal Melbourne Philharmonic Choir and Orchestra, and the New Zealand First Party. These entities are also *genuine existents*. Further, if various types of legal, political, financial, cultural and recreational activities had not developed on Earth, these entities would not exist. Thus, these are all existent entities that depend on rational activities.

Yet the US Supreme Court, the Microsoft Corporation, the Royal Melbourne Philharmonic Choir and Orchestra, and the New Zealand First Party are not causal social constructs in Haslanger's sense. For one thing,

[12]I assume a common-sense ontology of spatio-temporal objects. It falls outside the scope of this paper to worry about challenges to this assumption.

these entities are not spatio-temporal.[13] While each is intimately connected with some collection of spatio-temporal entities (e.g., the US Supreme Court is intimately connected with the Supreme Court Justices), none can be identified with any specific spatio-temporal entity, or any collection, mereological fusion, or set of spatio-temporal entities. Indeed, in a recent paper, Gabriel Uzquiano [2004] argues that entities of this type are of a new, and as yet not well-understood, metaphysical category, which he calls groups.[14]

Uzquiano says little about groups other than to argue for their distinctness from collections, fusions, and sets. The essence of Uzquiano's argument is easily understood. Membership of one description or another—set membership, parthood, etc.—is of central importance to the identity conditions of collections, fusions, and sets, while social factors of one description or another are more important than membership to the identity conditions of groups. Uzquiano gives no general characterization of the type of social factors that are central to the identity conditions of groups—he is not even explicit that it is always social factors that are central to these identity conditions—but rather makes specific claims such as, 'in the case of the Supreme Court, permanence of powers, rules, and procedures matters to the Supreme Court's continued existence more than permanence of members does' [Uzquiano 2004: 150].

Why, then, do I claim that it is social factors of one description or another that are of central importance to the identity conditions of groups? Let us begin by considering the US Supreme Court, and, to help us characterize its identity conditions, let us ask, What is responsible for the Supreme Court's existence? It exists because, as a social group, citizens of the USA grant members of that court certain rights, responsibilities, entitlements, and authorities. For example, its members are collectively authorized to interpret and uphold the integrity of the US Constitution, collectively have the right to challenge certain legislation, and are entitled to a certain type of legal respect. It is the continued presence of these rights, responsibilities, entitlements, and authorities, *and* the mechanisms for implementing them, that are most central to the continued existence of the US Supreme Court. These rights, responsibilities, entitlements, and authorities are also the factors that are central to its identity conditions.

Next, consider the Microsoft Corporation. While its primary influence on the world is that of producing computer software, it is a legal entity. It is an entity that has the right to, and assumes the responsibilities of being able to, participate in legal contracts. The Microsoft Corporation could not exist without the rationally constrained activities governing US corporations. Thus, the Microsoft Corporation—as opposed to its employees, buildings, products, etc.—exists in virtue of a certain social group—acting through the US Government—having invested certain individuals—roughly speaking, the Board of Directors of Microsoft—with certain rights, responsibilities,

[13]A later example, GeneriChem, constitutes the strongest evidence that I present for this claim and several others I make in the next couple of paragraphs.
[14]In this article, Uzquiano makes no mention of earlier philosophical discussions of these types of entities (see, e.g., Quinton [1976] and Mellor [1982]).

entitlements, and authorities. Further, whether some corporation *is* the Microsoft Corporation is a matter of whether that corporation is legally entitled to do things that only the Microsoft Corporation is entitled to do, and whether that corporation should be held legally responsible for the things for which only the Microsoft Corporation should be held responsible. That is, the identity conditions of the Microsoft Corporation should be specified in terms of the rights, responsibilities, entitlements, and authorities that are constitutive of its existence. Similar remarks can be made about the Royal Melbourne Philharmonic Choir and Orchestra and the New Zealand First Party.

While these familiar examples give a sense of the importance of social factors in determining the nature and identity conditions of groups, they do not fully illustrate this importance. To do so, consider a fictitious example that describes two groups of a kind that could arise—and undoubtedly have—in real life. Suppose that GeneriChem is a corporation that commercially produces industrial chemicals at dozens of plants. Periodically, industrial accidents happen at GeneriChem plants. When such accidents occur, the Board of Directors of GeneriChem sends an investigative team to determine what caused the accident and what changes, if any, might be needed in the operating procedures of GeneriChem plants. Company policies determine that such teams consist of one senior scientist, who is permanently employed by GeneriChem, and three other individuals who are, typically, employed by GeneriChem as consultants for the duration of a particular investigation. Formally, such teams exist for the exact period of time that the four individuals are at a particular plant performing a particular investigation. The senior scientist is responsible for coordinating such investigations and for writing an appropriate report once these investigative teams are disbanded.

Now, an industrial accident occurs at one of GeneriChem's plants that is more serious than any previous accident. As a consequence, the Board of Directors does not want to entrust this particular investigation to just one of its senior scientists; it would prefer to send two. Company policy mandates that each of the senior scientists on such an investigation must be a team leader, but experience has shown that sending investigative teams of more or less than four people to accidents tends to result in lack of cooperation or insufficient data from the plant under investigation. So, rather than sending two disjoint teams of four—essentially, an investigative team of eight—the Board of Directors decides to send the following four individuals: SS1, SS2, CS1, and CS2, where SS1 and SS2 are senior scientists. By stipulation, these four individuals will constitute *two* investigative teams: Team 1, led by SS1 and containing team members SS2, CS1, and CS2, and Team 2, led by SS2 and containing team members SS1, CS1, and CS2. At the investigation's conclusion, both SS1 and SS2 will write and submit independent reports. Now, if we stipulate that neither SS1 nor SS2 takes any action that is specific to writing his or her report while he or she is 'on-site', it is clear that we can further stipulate that every particular action taken by each of these four individuals while these two teams are in existence counts equally toward his or her memberships in Team 1 and Team 2. In other words, there is no way to physically distinguish these two teams. That there are two investigative teams—rather than just one—in existence during this investigation can only be accounted for by acknowledging the social authority of the Board of Directors in constituting and determining the identity conditions of its investigative teams.

Let us say that an item is a *constitutive social construct* if and only if it exists in virtue of a group of individuals having granted some item a normative role in certain of their activities.[15] For example, some individual or collection of individuals could be granted certain rights, responsibilities, entitlements, or authorities relating to the activities in question, or a statement or collection of statements could be granted the role of a rule or normative constraint on the activities in question. Groups, in Uzquiano's sense, are constitutive social constructs. What matters to their continued existence more than their membership is the continuation of the rights, responsibilities, entitlements, or authorities that are responsible for their existence, and the social activities that surround their members exercising those rights, responsibilities, entitlements, or authorities. Further, while the groups that I have mentioned have many members, groups—or constitutive social constructs very similar to groups—can have just one. The Prime Minister of New Zealand—the institutional role, as opposed to Helen Clark, the individual who occupies that role at the time at which this paper is being written—is an example.

Groups are not the only constitutive social constructs, however. Consider the rules of Texas Hold 'Em poker. A certain collection of statements is the rules of Texas Hold 'Em in virtue of the role that certain individuals grant those statements when engaged in certain card playing activities. These rules, then, are a constitutive social construct, just not one intimately related to some specific spatio-temporal entity or non-trivial collection of spatio-temporal entities. They are, rather, intimately related to an abstract entity, viz., a particular collection of statements. Yet even this link to some entity or other is not required in order for a constitutive social construct to exist. Consider, for example, those rule-governed activities in which one can engage in a purely intellectual manner, such as the game of chess. The game of chess exists wholly in virtue of certain individuals engaging in activities governed by the rules of that game. While the *rules* of chess are *important* to the existence of chess, the game of chess is *not identical with* its rules. Constitutive social constructs like chess, i.e., constitutive social constructs that are not related in this specific way to some entity or collection of entities, will be important to us in what follows. Let us call them *pure constitutive social constructs*.[16]

Further, note that constitutive and causal social constructs are not mutually exclusive. Many cases of social construction involve both constitutive and causal elements, though one or the other might be

[15] I borrow the term 'constitutive social construct', though not this definition, from Sally Haslanger (see, e.g., [1995]).

[16] The *contrast class* of pure constitutive social constructs can be characterized in the following way: an entity or collection of entities exists prior to the constitutive acts responsible for such a constitutive social construct's existence, and this entity or collection of entities is endowed with some additional feature or features by the constitutive acts in question.

dominant in any particular case. An example of this is a regulation baseball for US Major League play. Two distinct types of considerations are involved in something's being a regulation baseball. First, the ball in question must have certain physical characteristics (e.g., it must be a certain size, shape, and colour). Since regulation baseballs are manufactured to have these characteristics, these baseballs are causal social constructs. The second consideration is that the ball has to have been deemed regulation by an individual acting on behalf of the League and be signed by the League's Commissioner. This consideration makes regulation baseballs constitutive social constructs. A similar situation surrounds a piece of paper's being a twenty-pound note. More is required than the Bank of England putting it into circulation—roughly speaking, the activity that makes it a constitutive social construct. The paper must be of a certain quality, have a certain shape, and have a certain design printed on it with a certain type of ink. Twenty-pound notes are causal social constructs, because De La Rue plc manufactures them to have these features.

4. A Solution to the Puzzle

We began our discussion of modes of dependence and social construction with the aim of specifying a solution to a puzzle concerning the metaphysical interpretation of mathematical theories. Let us now employ it for this task. I noted above that securing the dependence of mathematical entities on mathematical activities would suffice to accommodate the creative freedom mathematicians feel they enjoy while engaging in mathematical theorizing. We must now ask whether this dependence should be taken to be solely causal, solely constitutive, or a combination.

Reflection points towards taking the dependence in question to be solely constitutive. Above, we saw good reasons for denying that mathematical entities are spatio-temporal entities. If we accept these reasons, then mathematical entities are not, in a strict sense, causal entities; that is, they do not stand in strict causal relations to other entities. One might say that certain legal and political activities 'cause' there to be a US Supreme Court. Such 'causation' is not strict in the sense in which I am using this term. Any type of causal dependence of mathematical entities on mathematical activities would require mathematical entities to stand in strict causal relations to mathematical activities.

Additionally, a complete solution to our puzzle requires that we vindicate the authority that mathematicians feel they have over certain matters to do with the ontology of mathematics.[17] We have seen that constitutive social construction is frequently linked with the granting of certain types of authorities and entitlements. The third observation at the beginning of this paper can thus easily be validated by the thesis that *mathematical entities are constitutive social constructs*. It can easily be argued that, just as members of the US Congress have the authority to constitutively construct the legal

[17]Recall that the truth of certain existential pure mathematical statements has to be discovered.

statutes of the USA, and representatives of the Major League have the authority to constitutively construct regulation baseballs, so, too, mathematicians have the authority to constitutively construct mathematical entities. Of course, there is a difference between the authority of mathematicians and members of Congress and representatives of the Major League; the authorities and privileges of members of Congress and representatives of the Major League are explicit and formally endowed, while those of mathematicians are implicit and only informally endowed. In explicating constitutive social construction, I have concentrated on explicit, formal instances of it. Yet it is clear that implicit, informal instances of this type of social construction abound. Many collectives (e.g., grade school cliques, siblings, and exercise partners) have leaders whose authorities and entitlements are never *explicitly* articulated. Their leadership is apparent, however, in the patterns of deference in operation within the collective and the role that these leaders play in interactions with those not in the collective. Despite the informal, implicit nature of their status, the authority of such leaders frequently extends to include the power to constitutively construct rules for the collective. Patterns of deference similar to these exist between mathematicians and the human population as a whole when it comes to mathematical matters. One important aspect of this—the deferential relationship between mathematicians and the scientific community—has been widely recognized in the recent literature. It is mathematicians' sensitivity to this implicit authority that is responsible for the almost absolute authority they feel they enjoy over certain matters concerning the ontology of mathematics.

At this point, a moment's reflection should make it clear that a plausible resolution of our puzzle is to maintain that

> *mathematical entities are* ***pure*** constitutive social constructs ***constituted by mathematical activities***.

For it is clear that there are no entities that are independent of mathematical activities to which mathematical activities can be interpreted as merely adding some feature or features. Further, this thesis makes it perfectly appropriate to endorse existential pure mathematical statements and refer to mathematical entities, for constitutive social constructs genuinely exist. Also, mathematicians really do have the creative freedom they feel they enjoy when engaging in mathematical theorizing. Indeed, mathematical entities constituted by mathematical activities vindicate the creative nature of mathematical postulation without jeopardizing its relative freedom. In addition, this thesis allows us to secure mathematicians' authority over certain matters concerning mathematical ontology in the way mentioned in the previous paragraph.

The thesis that mathematical entities—specifically mathematical domains—are pure constitutive social constructs constituted by mathematical practices, i.e., the rationally constrained social activities of mathematicians, is the metaphysical heart of an interpretation of mathematical theories that I call *Practice-Dependent Realism* (PDR). I take PDR to be a form of 'realism', because it takes mathematical entities to exist and there to be objective truths about these entities.[18] I take this realism to be 'practice-dependent', because the existence in question depends on mathematical practices. Officially, PDR is the conjunction of three theses about mathematical domains:

1. At least some exist;
2. they are and are made up of abstract entities;
3. they are constituted by mathematical practices as pure constitutive social constructs.

5. Practice-Dependent Realism: Some Details and Philosophical Merits

A number of aspects of PDR call for further explication. Perhaps the most natural question to ask is, What feature or features of mathematical practices is or are responsible for constituting mathematical domains? It is well known that new mathematical theories develop informally; well before there are formal axioms characterizing a new mathematical subject matter, mathematicians characterize it informally. These informal characterizations are assisted by the giving and accepting of informal proofs concerning the subject matter in question. Being explicit about which moves are acceptable and which are unacceptable, particularly when the moves involve statements that could be formally represented using universal and existential quantifiers, helps to provide a specification of the *ontological structure* of the subject matter in question. That is, being explicit helps to provide a specification of which objects make up that subject matter, what properties those objects have, and what relations those objects stand in to one another. As these informal practices develop, formal axioms can be introduced that—ideally—provide a categorical axiomatization of the subject matter in question, i.e., an axiomatization that specifies a unique structure up to isomorphism, where a *structure* is a collection of objects that have determinate properties and stand in determinate relations to one another. According to PDR, it is the incorporation of characterizations of new subject matters that is primarily responsible for the constructive nature of mathematical practices. A mathematical practice is constitutively responsible for the existence of a mathematical domain if and only if it includes a characterization of the ontological structure of that domain.[19]

In assessing the last statement according to PDR, the reader should be careful to remember that, as with all constitutive social constructs, a mathematical domain exists only if there is some non-trivial collection of

[18]See §5 for justification.

[19]Only some mathematical theories have a subject matter in this sense. Roughly speaking, non-algebraic theories (e.g., number theory, complex analysis, and set theory) have such a subject matter, while algebraic theories (e.g., group theory, ring theory, and field theory) do not. According to PDR, it is the former type of theory that is *primarily* responsible for the constitutive construction of mathematical domains.

mathematicians who recognize its existence by participating in or legitimizing the practice responsible for its existence.[20] Thus, to understand the above statement correctly, a mathematical practice must be recognized to be a social object in the sense that it is a collection of normatively constrained activities participated in, or recognized as legitimate, by a non-trivial collection of mathematicians; a single individual cannot be responsible for a mathematical domain existing unless his or her work in characterizing it is—in some sense—legitimized by a broader mathematical community. This need not exclude 'Crusoean' mathematicians—or the like[21]—from being responsible for the existence of mathematical domains, however, for the relevant type of legitimacy might be granted by the mathematical community to anyone who employs an appropriate methodology with an appropriate collection of representational/logical tools. A precise PDR-based characterization of this social aspect of mathematical existence falls outside the scope of this paper. Yet, this social aspect to the metaphysical account of mathematical domains offered by PDR is worth mentioning and remembering.

The above discussion of how mathematical practices are responsible for mathematical domains existing should assist the reader in answering a question that is likely to have occurred to him or her: Why does PDR concern domains rather than individual mathematical entities? The simple answer is that a mathematical practice only has the ability to characterize a structure in its entirety. Further, it would be *ad hoc* to claim that a mathematical practice was responsible for constituting only *some* of the entities in a structure it characterizes. Therefore, PDR maintains that a mathematical practice constitutes all of the entities in a structure it characterizes as a completed totality, together with their properties and relations. This is precisely what a mathematical domain is.

This discussion of the constitution of mathematical domains by mathematical practices also highlights a secondary advantage of PDR. In recent years, structuralist interpretations of mathematics—interpretations that, roughly speaking, take the subject matter of mathematics to have only structural features rather than both structural and non-structural features—have become extremely popular. PDR provides a natural account of the correctness of this insight. Since the features of mathematical practices responsible for constituting mathematical domains are features that only have the ability to characterize the structural features of mathematical entities, PDR predicts that the subject matter of mathematics is purely structural in nature.

A further characteristic of PDR that warrants additional explication is thesis 2, i.e., mathematical domains are and are made up of abstract entities. According to many traditional conceptions, abstract entities are paradigm

[20]Officially, this non-trivial collection must exist in the actual world in some region of space-time. This region of space-time need not occupy what we intuitively think of as the past or present. I cannot fully address PDR's pronouncements on the temporal and modal profile of mathematical domains in this paper, but will briefly discuss them later in this section.

[21]For example, Gauss' work on non-standard geometries might have been responsible for bringing certain geometric domains into existence despite the fact that he never shared that work with his contemporaries. I thank Neil Tennant and Roy Cook for focusing my attention on these types of cases.

cases of entities that are independent of rational activities. Consequently, the suggestion that certain pure constitutive social constructs are abstract might strike some as absurd. However, as I see things, 'abstract' is a philosophical term of art. While its primary uses share something in common—they all contrast abstract entities with concrete, most importantly spatio-temporal, entities—its precise use varies from philosopher to philosopher. The account of 'abstract' that I favour—because I think it does the best job of accommodating these differences in use—is one according to which *abstract* is a cluster concept, i.e., a concept whose application is marked by a collection of other concepts, some of which are more important to its application than others. The most central members of the cluster associated with *abstract* are the following:

1. *non-spatio-temporality*—the item does not stand to other items in the spatio-temporal relations characteristic of being a constituent of the spatio-temporal world;
2. *acausality*—the item neither exerts a strict causal influence over other items nor does any other item causally influence it in the strict sense;
3. *eternality*—the item exists timelessly; and
4. *changelessness*—none of the item's intrinsic properties change.[22]

An item is *abstract* if and only if it has a sufficient number of these features.

According to PDR, mathematical domains are constituted to have all four features. I cannot defend even the possibility of this being true in an adequate manner in this paper. Yet let me provide some sense of how this could be. The key is that constitutive social constructs can be constituted to have spatio-temporal and causal features that are quite different from the activities, decisions, and practices that constitute them. For example, legal and political borders are constituted with spatio-temporal features different from the legislative and political activities that constitute them. Those activities take place in a particular location, during a particular period of time. Generally, the borders they constitute have a different location and only come into existence at some time after the activities that are responsible for their existence end. Further, as causally instantiated activities, all acts, decisions, and practices have a range of innocuous strict causal features (e.g., the expenditure of energy). Yet most constitutive social constructs lack the majority of such innocuous strict causal features. Now, consider the property 'being on the official injury list of the Major League'. This property

[22]Roughly speaking, the intrinsic properties of an item are those that it has independently of its relationships to other items. While the intrinsic-extrinsic distinction is extremely controversial and tremendously difficult to draw in any clear way, this modifier—or a similar one—is needed here, because it is clear that the extrinsic properties of mathematical entities can change. For example, the extrinsic properties of the number 7 would change were I to decide that it is no longer my favourite natural number.

is bestowed upon an individual for a particular period of time by representatives of the Major League. Thus, it is a constitutive social property. Said representatives frequently bestow it on individuals retroactively (e.g., on a Friday, these representatives stipulate that a certain player has been on this list since Tuesday). So, at least certain constitutive social facts about the past are, to use the standard metaphysical terminology, *soft*, i.e., they are the kind of facts about the past that can be changed by events, activities, etc. in the present or future.

Given these observations about the causal and spatio-temporal features of constitutive social constructs, it seems reasonable to maintain that we could constitute a pure constitutive social construct that has neither causal features nor the spatio-temporal features characteristic of being a constituent of the spatio-temporal world. Indeed, my last example opens the possibility that we could constitute a pure constitutive social construct that has always existed and always will or at least that has temporal characteristics quite distinct from those of the practice responsible for its existence. Further, if it is possible to constitute a pure constitutive social construct that has the first three features in the cluster associated with *abstract*, it is difficult to believe that it is not possible to constitute such an item with the added feature of being changeless. After all, the causal features of items are the primary source of changes in those items' intrinsic properties. So, as I understand *abstract*, constitutive social constructs can be abstract entities. Indeed, I think reflection will reveal that, at least intuitively, many constitutive social constructs are.

Let us next turn to my claim that, according to PDR, there are objective truths about mathematical entities. As the objectivity of mathematics is a primary reason why some are tempted to advocate realism or Platonism, the reader might have greeted this claim with scepticism. To assuage this scepticism, consider the following: it is a wholly objective truth about the US Supreme Court that, when full, it has nine members. And it is a wholly objective truth about the political border between the USA and Canada that, as of this date, it runs through Lake Superior. So, in some sense, there are objective truths about constitutive social constructs. Consequently, if PDR is correct, the fact that mathematical domains are constitutive social constructs does not exclude there being objective mathematical truths. The trick, of course, is to identify the sense in which at least some mathematical truths are objective.

Perhaps the best place to start is with a general characterization of objectivity. Intuitively, a statement P is objectively true in the epistemic sense if and only if P is true independently of all rationally constrained judgments relating to the subject matter of P.[23] The examples in the last paragraph show that there is a problem with this general characterization. While intuitively both statements are objectively true, neither is true independently of all rationally constrained judgments concerning its subject

[23]See Searle [1995] for a characterization of the distinction between epistemic and metaphysical notions of objectivity.

matter. The first depends on the rational activities that constitute the US Supreme Court, while the second depends on the rational activities that constitute the political border between the USA and Canada.

What, then, are we recognizing when we claim that these truths are objective? Roughly speaking, we are recognizing that there are determinate facts about the constitution of the US Supreme Court and the political border between the USA and Canada that make these statements true. Given this, we should all acknowledge their truth, for unless we happen to be engaged in activities that have the social authority to change these constitutive facts, we do not have the authority to dissent from the truth of these statements—it is not under our control and cannot be legitimately denied.

This discussion suggests a modified characterization of epistemic objectivity: a statement *P is objectively true* if and only if P is true independently of all rationally constrained judgments relating to the subject matter of P *other than* any judgments that play a constitutive role in the constitution of the subject matter of P when the subject matter of P is a, perhaps trivial, collection of constitutive social constructs. This definition collapses into the intuitive definition of objectivity given above for subject matters that are independent of rationally constrained activities, and allows for the existence of objective truths about constitutive social constructs, which the above examples indicate there are.

Now, according to this modified definition, are there objective mathematical truths? Quite clearly, there are. Let P be a mathematical statement whose truth value is determinately specified by the activities that constitute mathematical domains (e.g., all widely accepted mathematical axioms). Then P's truth value is determined by judgments that play a constitutive role in the constitution of the subject matter of P, and is independent of all other judgments. Consequently, according to this account of objectivity, P is either objectively true or objectively false, for its status as one or the other is secured by the determinateness and constitutive authority of the relevant constitutive judgments. So, all socially acceptable mathematical axioms are objectively true according to this account of objectivity. Further, if we acknowledge that logical consequence is objective, then so are all logical consequences of these axioms.[24]

Importantly, the observations of the last paragraph accord well with a widely acknowledged feature of mathematicians' debates about the objectivity of certain mathematical statements (see, e.g., Balaguer [1998]). Consider Cantor's continuum hypothesis (*CH*)—the set-theoretic statement

[24]Given the links between set theory and logical consequence, questions may legitimately be asked about whether proponents of PDR can simply exploit the objectivity of logical consequence in their explanation of the objectivity of mathematics. One strategy that such a proponent could use would be to follow certain fictionalists, modal nominalists, and non-traditional Platonists in rejecting the standard set-theoretic account of logical consequence and instead insisting on one which exploits a modal primitive. I am not aware of any such account that does justice to the undeniable interplay between mathematics and our understanding of logical possibility, yet such accounts point in the direction of the kind of explanation of the objectivity of logical consequence that I favour.

that there is no set of real numbers whose cardinality is greater than that of the natural numbers yet less than that of all real numbers. Debates about whether this and similar set-theoretic statements have objective truth values centre around the strength and determinateness of our concept of set, not whether a Platonistically construed set-theoretic hierarchy exists. Specifically, the debate about whether CH has an objective truth value focuses on whether our concept of set is strong and determinate enough to yield an axiom that has CH or $\neg CH$ as a logical consequence.

Now, note that mathematicians' concept of a particular mathematical subject matter is precisely what delimits the constitutive judgments of the practice surrounding that subject matter. So, in effect, debates about the strength and determinacy of mathematicians' concept of a mathematical subject matter are debates about which statements are to be determinately accepted as constitutive judgments concerning that subject matter. Thus, if my modified definition of epistemic objectivity is correct, set-theorists are debating precisely what they should be in order to determine whether certain set-theoretic statements are objectively true.

In the preceding paragraphs, I have sketched an account of the objectivity of mathematical statements that conforms well with actual debates about their objectivity. Yet there is one intuition about the objectivity of certain mathematical statements with which I have not shown it to be compatible: that certain mathematical truths (e.g., arithmetical truths) are objective in a richer sense than certain other mathematical truths—roughly speaking, truths concerning extremely abstruse mathematical subject matters that have little or nothing to do with features of the spatio-temporal world. This richer sense of objectivity, the intuition goes, attaches to the first class of mathematical truths in virtue of their being in some way grounded in objective features of the spatio-temporal world, a world that is independent of our activities.

I think that there is something to this intuition, and it is perfectly compatible with PDR. To see this, let us first ask after the precise relationship between the mathematical theories covered by this intuition and the relevant features of the spatio-temporal world. As a starting point, consider arithmetic. Everyday arithmetic practice supports the truth of all specific instances of the following schema: the number of Fs is equal to n if and only if there are n Fs.[25] This schema states an important link between the natural numbers—a particular collection of mathematical entities—and cardinality properties of collections. For example, the collection of electrons orbiting the nucleus of a sodium atom has cardinality eleven. Specifically, natural numbers provide a first-order way of indirectly representing—and, consequently, more easily reasoning about—cardinality properties of collections.[26] After recognizing this feature of natural numbers, the reader should note that it would have provided—in fact, did provide—an excellent

[25]Neil Tennant discusses this schema extensively (see [1987] and [1997]).

[26]A first-order way of representing is one whose variables have objects in their range rather than properties or relations. A theory X—or, more loosely, the items that the variable of X range over—aids in the *indirect* representation of some item or collection of items Y if and only if it aids in the representation of Y and the item or items in Y do not fall within the range of the variables of X.

motivation for the constitutive construction of the domain of natural numbers. As another example, consider Euclidean geometry. Again, it is a tool for indirectly representing certain spatial relationships in a first-order way. In this case, there is good evidence that its introduction was motivated by the indirect representational benefits it provides, particularly with respect to the distribution of land.[27]

These examples show that the mathematical theories covered by the intuition under consideration provide us with tools for indirectly representing certain features of the spatio-temporal world in a first-order way. Thus, these mathematical theories are grounded in our ways of representing features of the spatio-temporal world. To restate, the representational relationships that these mathematical domains stand in to aspects of the spatio-temporal world determine that the mathematical theories that characterize these domains have a certain type of friction—in Kant's sense—with the spatio-temporal world.

These examples also make something else clear: the representational benefits that accompanied the introduction of the theories covered by the intuition under consideration were or could have been an important motivating factor for their introduction. It is this observation that makes it clear that PDR is compatible with the intuition under consideration. In order for PDR to be plausible, it must be that mathematicians had, and in some cases continue to have, at least *prima facie* good reasons for introducing mathematical theories that characterize or characterized previously uncharacterized mathematical domains. After all, at least in general, we do not socially constitute entities for no reason whatsoever. The representational benefits just identified constitute such reasons.[28]

There is one final topic that I wish to address: mathematical epistemology. The epistemological challenge to Platonism has been extremely influential in the philosophy of mathematics (see, e.g., Benacerraf [1973] and Field [1989]). Acceptance of some variety of this challenge has motivated many to deny the existence of mathematical entities. A full exploration of this challenge—showing that one of its assumptions is that mathematical entities exist independently of all rational practices—must be postponed for another paper. Yet, given that PDR does maintain that mathematical entities exist, it is worth briefly indicating why we should be optimistic that PDR has the resources to provide a positive account of mathematical knowledge.

Proponents of the standard epistemological challenge to Platonism assume that, in order for human beings to have knowledge of existent mathematical entities, those entities must, in some sense, influence us, i.e., there must be influence from the mathematical realm to the spatio-temporal

[27]Many standard histories of mathematics note that early geometric practices arose on the banks of rivers, and suggest that agricultural needs (e.g., land distribution) motivated their introduction (see, e.g., Eves [1976]).

[28]These representational benefits can also be used as the cornerstone of the *beginnings* of an account of the applicability of mathematics. See Yablo [2005] for further details on how this account might begin.

world. The most plausible existent responses to this challenge (e.g., Mark Balaguer's [1998] and Stewart Shapiro's [1997]) argue that no such contact is needed. A proponent of PDR can offer a radically different response to this type of scepticism: specifically, that mathematical practices determine, i.e., set, the features of mathematical domains—the features of mathematical domains are a projection of the features ascribed to them within mathematical practices. Further, since mathematical practices are spatio-temporally instantiated, they can causally influence us in ways that allow for our knowledge of at least some of the features of mathematical domains. In other words, a proponent of PDR can exploit a *constitutive* influence from the spatio-temporal world to the mathematical realm in his or her account of mathematical knowledge.

The force of this response can, perhaps, be more easily comprehended if it is illustrated in another context. One interesting collection of constitutive social constructs is the legal statutes of the USA. Let us briefly investigate how people gain knowledge of what legal statutes exist and what features they have. Put simply, it is by causally interacting with those activities and practices that are responsible for such legal statutes existing and having the features that they do. Typically, our causal interaction with these activities and practices is indirect. For example, we might learn these legal truths from classroom teachers, parents, or friends, read about them in newspapers, magazines, or on the Internet, or hear about them on a television or radio programme. Yet this is a legitimate mechanism for obtaining this type of knowledge, because, ultimately, such causal interactions can be traced to the acts, decisions, and practices that determined the features of such statutes and thus made these legal truths true.[29]

6. Conclusion

I must end my exposition of PDR at this point. Clearly, there are still many interesting aspects of this interpretation of mathematical theories that warrant further explication. For example, how might PDR account for the *prima facie* necessity of mathematics? Or its *prima facie* apriority? What account can PDR provide of the applicability of mathematics—both to the spatio-temporal world and internally to mathematics? Can the details of the above account of the objectivity of mathematics be filled in? How about PDR's positive proposal with respect to mathematical epistemology? Despite these many gaps, I hope to have provided enough details in this paper to warrant taking PDR seriously. The puzzle developed in §1 is, or at least should be, worrying to all who promote realist, Platonist, fictionalist, and modal nominalist interpretations of mathematical theories. So far as I can tell, the only plausible solution to this puzzle is to promote an interpretation of mathematics that is similar to PDR. Given this, and the

[29]There are deep parallels between legal knowledge and mathematical knowledge. Unfortunately, space constraints prevent further exploration of them in this paper.

philosophical merits of PDR, I recommend it as an optimal metaphysical interpretation of mathematical theories.[30]

Buffalo State College

Received: July 2007
Revised: June 2008

References

Balaguer, Mark 1998. *Platonism and Anti-Platonism in Mathematics*, Oxford: Oxford University Press.
Benacerraf, Paul 1973. Mathematical Truth, *Journal of Philosophy* 70/19: 661–79.
Burgess, John 2004. Mathematics and Bleak House, *Philosophia Mathematica* 12/3: 18–36.
Burgess, John and Gideon Rosen 2005. Nominalism Reconsidered, in *The Oxford Handbook of Philosophy of Mathematics and Logic*, ed. Stewart Shapiro, Oxford: Oxford University Press: 515–35.
Cole, Julian 2008. Mathematical Domains: Social Constructs? in *Proof and Other Dilemmas: Mathematics and Philosophy*, ed. Bonnie Gold and Roger Simons, Washington, D.C.: Mathematics Association of America: 109–28.
Dedekind, Richard 1888. *Was Sind und was Sollen die Zahlen?* Braunschweig, Germany: Friedrich Vieweg und Sohn.
Eves, Howard 1976. *An Introduction to the History of Mathematics*, 4th edn, New York: Holt, Rinehart and Winston Press.
Field, Hartry 1989. *Realism, Mathematics, and Modality*, Oxford: Basil Blackwell.
Hamilton, William 1844. Letter to John T. Graves, Esq., *Dublin's Philosophical Magazine* 25/3: 489–95.
Haslanger, Sally 1995. Ontology and Social Construction, *Philosophical Topics* 23/2: 95–125.
Hurwitz, Adolf 1898. Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*: 309–16.
Mellor, D. H. 1982. The Reduction of Society, *Philosophy: The Journal of the Royal Institute of Philosophy* 57/219: 51–76.
Quinton, Anthony 1976. Social Objects, *Proceedings of the Aristotelian Society* 76: 1–27.
Searle, John 1995. *The Construction of Social Reality*, New York: The Free Press.
Shapiro, Stewart 1997. *Philosophy of Mathematics: Structure and Ontology*, Oxford: Oxford University Press.
Tennant, Neil 1987. *Anti-Realism and Logic*, Oxford: Oxford University Press.
Tennant, Neil 1997. On the Necessary Existence of Numbers, *Noûs* 31/3: 307–36.
Uzquiano, Gabriel 2004. The Supreme Court and the Supreme Court Justices: A Metaphysical Puzzle, *Noûs* 38/1: 135–53.
Yablo, Stephen 2005. The Myth of the Seven, in *Fictionalism in Metaphysics*, ed. Mark Kalderon, Oxford: Oxford University Press: 88–115.

[30]I thank Stewart Shapiro for his support during the development of PDR and for extensive comments on a variety of papers relating to PDR. I thank Gabriel Uzquiano for suggesting that I consider a puzzle similar to the one discussed in §1 as a means of motivating PDR. I thank Neil Tennant for valuable insights into how to make the arguments and position developed in this paper perspicuous to a first time reader. I also thank Robert Batterman, Kimberly Blessing, Eric Carter, John Corcoran, Roy Cook, Salvatore Florio, Robert Kraut, Christopher Pincock, George Schuum, Lisa Shabel, Kelly Trogdon, attenders of the Seventh Annual Midwest Philosophy of Mathematics Workshop, attenders of the second meeting of the 2008–2009 Buffalo Logic Colloquium, and two anonymous referees for the *Australasian Journal of Philosophy* for comments on assorted drafts of this paper. Finally, I thank Barbara Olsafsky for her support, willingness to discuss my work with me, and readiness to proofread my papers.