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Mathematical Domains: Social Constructs?

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Social constructivism is probably the philosophy of mathematics that has seen the greatest growth in support among mathematicians in the last twenty-five years. However, until now, because of assorted difficulties that this view appears to imply, it has not received serious attention from philosophers of mathematics. For example, see Balaguer's quick dismissal of it in his chapter (section 2.2 and elsewhere). Julian Cole is the first philosopher to seriously attempt to deal with these problems. This chapter gives you an introduction to the philosophical issues and how he is attempting to deal with them. His view is still being developed. After reading this chapter, you may want to follow his future work (and that of those who respond to it). In particular, his upcoming article, "Ontology, Freedom, and Authority: A New Perspective on the Metaphysics of Mathematics" seems likely to be of interest.

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1 *Introduction*

There can be little doubt that mathematics is a social activity. Among other things, mathematicians often work together in groups, they frequently choose to work on problems because other mathematicians deem them important or difficult or worthy, they rely on other mathematicians to verify the correctness of their work, they present their work in public forums, more than one mathematician (or group of mathematicians) can work on the same problem, and mathematicians compete with each other for sparse funding. That mathematics is social in all of these senses—and several others—is uncontentious. In the last ten years, however, two books[1] have been published that advocate that mathematics is social in a much deeper—and correspondingly more controversial—sense than any of these. The authors of these books—respectively, Reuben Hersh, a professional mathematician, and Paul Ernest, a specialist in mathematics education—suggest that the *subject matter* of mathematics is social. More precisely, they advocate the thesis that the subject matter of mathematics—mathematical domains or structures—is constructed by or created by—quite literally brought into existence by—the social activities of mathematicians. This is a contentious thesis if ever there was one, as, on a standard interpretation, it implies there were, for example, no numbers until mathematicians invented them.

The details of Ernest's and Hersh's accounts of how the subject matter of mathematics is socially constructed[2] are quite different. Further, anybody who has read their books will be aware that both Ernest and Hersh have a much broader agenda[3] than the mere advocacy of the thesis that the subject matter of mathematics is socially constructed. Indeed, it would probably be accurate to say that the articulation and defense of this thesis are in many ways secondary to both authors' primary goals. Nonetheless, both Ernest and Hersh do promote this thesis.

I am fascinated by the suggestion that mathematical domains (structures)[4] are socially constructed. What I would like to do in this chapter is to explore this suggestion with the aim of making one version of it reasonably precise and evaluating its merits and weaknesses. I shall undertake the latter task by comparing it with Platonism, its best-known rival. Ideally, I would also compare this version of social constructivism with the two popular versions of Nominalism, Fictionalism and Modal Nominalism. Unfortunately, space does not allow. For similar reasons, I shall be unable to discuss all of the details of this account of mathematics.

2 Ernest's and Hersh's View of Mathematics

Let us begin by considering what Ernest and Hersh say about the social construction of **mathematical ontology—**the mathematical items that exist. The following quotes indicate Ernest's general position about mathematical objects:

> According to the social constructivist view the discourse of mathematics creates a cultural domain within which the objects of mathematics are constituted by mathematical signs in use. ([Ernest 1998], p. 193)

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[1] Reuben Hersh's *What is Mathematics, Really?* [Hersh 1997] and Paul Ernest's *Social Constructivism as a Philosophy of Mathematics* [Ernest 1998].

[2] I shall provide an extensive discussion of social construction in Section 3.

[3] This broader agenda is, as a matter of fact, quite similar for both authors. They are both interested in exploring philosophical issues concerning mathematics using a much broader range of mathematical examples than is typical in the contemporary (analytic) philosophy of mathematics literature. They are both interested in discussing informal mathematical activities, e.g., the nature of mathematical intuition and how mathematical theories are generated and refined before they are formalized. (Discussions of this type are almost completely lacking in contemporary analytic philosophy of mathematics.) Additionally, they are both interested in combating what they believe—I think mistakenly—is a dominant conception of mathematical knowledge as certain and infallible.

[4] From this point onwards I shall simply talk about mathematical domains. I leave it open whether some or all mathematical domains are structures.

> . . . signifiers have ontological priority over the signified—especially in mathematics, for the signifiers can be inscribed and produced, or at least instantiated, whereas the signified can be indicated only indirectly, mediated through signifiers.
>
> ([Ernest 1998], p. 196)

> . . . the ontology of mathematics is given by the discursive realm of mathematics, which is populated by cultural objects, which have real existence in that domain . . . mathematical discourse as a living cultural entity creates the ontology of mathematics.
>
> ([Ernest 1998], p. 202)

While there is much in these quotes that the reader is likely to find perplexing and in need of further explanation, two points can be gleaned from them. First, Ernest's belief that mathematical objects are constructed by or created by—made real by—the activities of mathematicians. This is the **basic thesis of social constructivism** (about mathematics). Second, Ernest takes the discursive elements of mathematics to be central to the construction of mathematical ontology. Indeed, the first and second quotes indicate that Ernest believes that the constructive work of mathematical practices[5] is done, at least primarily, by the presence of mathematical signs and signifiers in the discursive elements of those practices.

The most natural interpretation of "mathematical signs" and "signifiers" in Ernest's quotes is one according to which they are **lexical items**—such items as the marks written down by mathematicians and the sounds uttered by mathematicians. Yet, under this interpretation, Ernest's suggestion is problematic. Mathematical discursive practices only contain a finite number of such signs and signifiers. Thus, if such signs and signifiers are responsible for the existence of all mathematical entities, then some of them must be responsible for the existence of collections of mathematical entities with infinite—indeed, extremely large infinite—cardinalities. *How* can they be so responsible? At least to my knowledge, Ernest has not provided an answer to this question.[6] So, Ernest's discussion of the social construction of mathematical ontology is unhelpful with respect to a key aspect of that construction. We shall explore this issue further in Section 4.

Let us investigate whether Hersh can provide us with a more helpful account of mathematics. Here are some quotes from his book:

> Fact 1: Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life.
>
> Fact 2: Once created, mathematical objects can have properties that are difficult for us to discover. ([Hersh 1997], p. 16)
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> 4. Mathematical objects are a distinct variety of social-historical objects. They're a special part of culture. ([Hersh 1997], p. 22)

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[5] A **practice** is a collection of activities governed by standards of correctness and incorrectness. A practice is **discursive** if the activities in question are ones that center about **assertoric content**, i.e., the thing that we can assert, assume, consider, etc. Many mathematical activities are discursive practices because they involve assertions, proofs, etc. For a detailed discussion of assertoric content, I refer the reader to Crispin Wright's discussion in Chapters 1 and 2 of [Wright 1992].

[6] Interestingly, non-mathematicians tend to find the criticism I level at Ernest in this paragraph obvious, while mathematicians sometimes have difficulties understanding the problem. I suspect that this is because mathematicians are so used to representing infinite collections of entities with a finite number of symbols that they intuitively fill in an answer to my question. My point is simply this, while there most certainly is an answer that can be provided, Ernest has failed to provide it in his book.

In Fact 1, Hersh expresses the basic social constructivist thesis with a minor twist: he recognizes the need to account for why human beings created mathematical domains[7] and hints at such an account. In Fact 2, Hersh indicates his sensitivity to a certain type of independence that mathematical domains have from mathematical practices—let us call it **epistemic**[8] **independence**, for it relates to our knowledge of mathematical domains. Just below Fact 2, he tells us

> Once created and communicated, mathematical objects are *there*. They detach from their originator and become part of human culture. We learn of them as external objects, with known and unknown properties. Of the unknown properties, there are some that we are able to discover. Some we can't discover, even though they are our own creations.
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> ([Hersh 1997], p. 16)

The second part of this quote reinforces Hersh's sensitivity to the epistemic independence of mathematical domains from mathematical practices. The first part of this quote goes further than this, however. It indicates that mathematical domains detach—in some sense—from their specific creator. We shall return to this point in Section 4.

Perhaps Hersh's most interesting claim, however, is that "mathematical objects are . . . social-historical objects" ([Hersh 1997], p. 22). What are we to make of this claim? I believe that the following quote is helpful:

> Frege showed that mathematical objects are neither physical nor mental. He labeled them "abstract objects." What did he tell us about abstract objects? Only this: They're neither physical nor mental.
>
> Are there other things besides numbers that aren't mental or physical?
>
> Yes! Sonatas. Prices. Eviction notices. Declarations of war.
>
> Not mental or physical, but not abstract either!
>
> The U.S. Supreme Court exists. It can condemn you to death!
>
> Is the court physical? If the Court building were blown up and the justices moved to the Pentagon, the Court would go on. Is it mental? If all nine justices expired in a suicide cult, they'd be replaced. The court would go on.
>
> The Court isn't the stones of its building, nor is it anyone's minds and bodies. Physical and mental embodiment are necessary to it, but they're not *it. It's a social institution*. Mental and physical categories are insufficient to understand it. It's comprehensible only in the context of American society.
>
> What matters to people nowadays?
>
> Marriage, divorce, child care.
>
> Advertising and shopping.

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[7] The observant reader will have noticed that both Ernest and Hersh talk about the construction of mathematical objects while I talk about the construction of mathematical domains. There are two reasons for this. First, it seems to me that (at least most) mathematical objects are the objects they are in virtue of their relationships to the other objects in the domain of which they are a member. So, in order to construct a particular mathematical object, one really needs to construct all of the objects in the domain of which that object is a member. Second, in constructing some aspect of mathematical reality, one is presumably not only constructing the objects in that aspect of mathematical reality, but also the properties of those objects and the relationships between those objects. A domain, at least as I am using this notion, is a collection of objects that have properties and stand in relations to one another.

[8] **Epistemology** is the branch of philosophy that investigates the nature of knowledge and justification.

Jobs, salaries, money.
The news, and other television entertainment.
War and peace.
All these entities have mental and physical aspects, but none is a mental or a physical entity. Every one is a social[-historical] entity. ([Hersh 1997], pp. 13–14)

In this passage, Hersh mentions a wide variety of social-historical entities, some legal (e.g., eviction notices and the U.S. Supreme Court), some political (e.g., declarations of war and peace), some financial (e.g., money and salaries), and others recreational (e.g., sonatas and television programs). All of these items exist, and their existence has very real consequences. Yet they owe their existence to the power of certain types of acts, decisions or practices undertaken by human beings to make certain items real simply by happening or being undertaken. In suggesting that mathematical entities are social-historical entities, Hersh is suggesting that the same is true of mathematical domains. That is, mathematical domains exist and they owe their existence to the power of certain mathematical activities undertaken by human (and other rational) beings to make them real simply by being undertaken. Let us call this the **social-institutional understanding of the nature of mathematics**.

In what follows, when I talk about mathematical domains as social constructs, I shall have in mind Hersh's social-institutional understanding of the nature of mathematics. I believe that it is more promising than Ernest's signifier-signified understanding of mathematics. While many of the practices that constitute social-historical entities involve signs and signifiers, the presence of these signs and signifiers is not, in general, central to these practices' constructive power. The above discussion of Ernest's account of mathematics certainly suggests that, if the basic social constructivist insight is correct, then the same is true in the mathematical case.

3 Social Construction and Dependence

In this section, I provide one framework for how objects come to be socially constructed. I do this so that I can locate the social construction of mathematical domains within this framework. In "Ontology and Social Construction" ([Haslanger 1995]), Sally Haslanger gives expression to a variety of ways in which social acts, decisions, or practices might be involved in social construction. The most basic distinction she makes is that between "causal social construction" and "constitutive social construction." These are two ways of constructing existent items. Haslanger offers the following characterizations of these two varieties of social construction:

Causal social construction: Something is causally socially constructed if[9] social factors play a *causal* role in bringing it into existence or, to some substantial extent, in its being the way that it is.

Constitutive social construction: Something is constitutively socially constructed if a *correct definition* or account of what it is for something to be an item of the type in question *must make reference to social factors*.[10]

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[9] Throughout I follow the mathematical convention of leaving 'only if' out of definitions.

[10] These definitions are taken from page 98 of [Haslanger 1995], though I have slightly modified the second.

Consider first such items as cars, scissors, alarm clocks, and telephones. These are spatio-temporal entities that have been manufactured for some particular purpose. Let us call such items **artifacts**. Clearly, artifacts would not exist if there were no social acts, decisions, or practices. So, artifacts are dependent on[11] certain social acts, decisions, or practices. Further, the primary mechanism of artifacts' dependence on social acts, decisions, or practices is well understood. Artifacts are *causally dependent* on the social acts, decisions, or practices that bring them into existence. Thus, artifacts are **causal social constructs**—the products of causal social construction.

Next, consider the examples that Hersh mentions in the long quote in Section 2, and such items as legal borders between pieces of property (land), political borders between countries, property itself, countries themselves, laws (in the sense of statutes),[12] and games like baseball and tennis. It should be uncontroversial that all of these items exist. Further, a moment's reflection should make it clear that if various types of legal, political, financial, cultural and recreational practices had not developed on Earth, then none of these items would exist. Thus, all these items are dependent on social practices. Yet the *mechanism* of these items' dependence on social practices is different from that of artifacts' dependence on social acts, decisions, or practices. Social practices need not causally manipulate previously existing spatio-temporal items in order to bring legal and political borders, countries, laws, etc. into existence. Rather, these items simply owe their existence to certain social acts. It is this type of a dependence of an item on a social act that is characteristic of constitutive social construction. This type of dependence ensures that social factors have to be talked about in a correct definition or account of what the item is. Thus, these items are **constitutive social constructs**.[13]

Constitutive social constructs *can* have influence over the spatio-temporal world and the spatio-temporal world can have influence over which items we construct constitutively. One only need reflect on the impact of declarations of war to recognize this. What our contrast emphasizes is that the *means* by which an item becomes a *constitutive* social construct is not causal in the strict sense characteristic of *causal* social construction.

While my exposition so far might suggest that constitutive and causal social construction are mutually exclusive, this is not the case. Many cases of social construction involve both elements, though one or the other might be dominant in any particular case. An excellent example of this is a "regulation baseball" for Major League play. Two distinct types of considerations are involved in something's being a regulation baseball. First, the ball in question must have certain physical characteristics, e.g., it must be a certain size, shape, color, etc. Regulation baseballs are manufactured to have these characteristics. Thus, regulation baseballs are causal social constructs. The second consideration is that the ball has to have been deemed regulation by an individual

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[11] Let us say that an item X is **dependent on an act, decision, or practice** Y if X would not exist if Y did not occur or exist. Additionally, let us say that an item X is **independent of an act, decision, or practice** Y if X would exist even if Y did not occur or exist.

[12] Whenever I talk about laws in this chapter, I shall be talking about laws in the sense of statutes rather than laws in the sense of laws of nature or the laws of probability.

[13] I have to confess a certain level of dissatisfaction with Haslanger's definition of constitutive social construction. This dissatisfaction is rooted in the fact that her definition obscures the importance of the particular mechanism of dependence of an item on social factors that my examples serve to illustrate. This mechanism of dependence is central to my own thought about constitutive social construction.

acting on behalf of the League and be signed by the League's commissioner. This consideration makes regulation baseballs constitutive social constructs.

Many acts of constitutive social construction are accompanied by acts of causal social construction, or provide already existing objects with additional features. For example, in composing a sonata, a composer will usually write a score. When declaring war, a country will usually produce a written proclamation of war. In legally dividing a single piece of land into two pieces of land, the owners of the two properties will usually either construct a barrier of some description to mark the division or divide the land using a natural barrier. Some acts of constitutive social construction *require* an accompanying object or act of causal construction. For example, a representative of the Major League can only deem a baseball regulation if it has certain physical characteristics. You don't have an eviction *notice*—as opposed to an eviction *order*—without the piece of paper on which the eviction order is written. Other acts of constitutive social construction do not require any kind of associated object. For example, in legally dividing a single property into two smaller properties, there is no need to place a barrier between the two properties, and in declaring war, there is no need to write a proclamation.

Let us call constitutive social constructs that do not require any kind of associated object **pure constitutive social constructs**, and those that do **impure constitutive social constructs**. Pure constitutive social constructs exist wholly in virtue of the undertaking of certain acts, decisions, or practices of social significance. Legal statutes are pure constitutive social constructs: roughly speaking,[14] a collection of statements has the property of being a legal statute wholly in virtue of its having appropriately proceeded through the process of approval and having been passed by a legitimate legislative authority.[15] Political borders are also pure constitutive social constructs. Roughly speaking, a certain line's marking a political border is wholly a matter of certain decisions made by relevant political groups; there is no need for such a border to be marked in any particular way.

With the above conceptual tools in place, let us refine the basic thesis of social constructivism (about mathematics) into the **central thesis of social constructivism** (about mathematics): *mathematical domains (and the items of which they are composed) are pure constitutive social constructs constituted by mathematical practices*. That is, particular mathematical domains (and the items of which they are composed) exist wholly in virtue of the undertaking of mathematical practices of a specific type. In the next section, we shall consider what specific type of mathematical practice is required. In the remainder of this chapter, when I talk about social constructivism, I shall be talking about this thesis, not the wider agenda of most social constructivists (see Footnote 3).

Social constructivism's advocacy of the dependence of mathematical domains on mathematical practices is what distinguishes it from all forms of Platonism. For our purposes, **Platonism** is the conjunction of three theses about mathematical domains: a) some exist, b) they (and the items

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[14] There are other considerations involved. For example, a statute must not be declared unconstitutional and it must not be overridden by later legislative activities. None of these further considerations undermine the claim that legal statutes are pure constitutive social constructs.

[15] It is part of the procedure for passing a federal statute that various written versions of it are produced, including the version signed by the President. Yet, after signing, should all these required written versions of the statue be destroyed—perhaps by a nuclear attack in the D.C. area—it would remain law without them. Consequently, the statue itself has a certain type of independence from all of its written versions; they are not required for its continued existence.

of which they are composed) are paradigm cases of abstract entities[16]—so, for example, they are acasual, non-spatio-temporal, eternal, and changeless, and c) they (and the items of which they are composed) are independent of all social acts, decisions, and practices—they would exist even if there were no social acts, decisions, or practices.

4 Logic and Ontological Structure

We now have a basic understanding of the account of mathematical domains offered by social constructivists—mathematical domains are socially constituted by mathematical practices. Yet we still lack an answer to one important question: *according to social constructivists, how, exactly, do mathematical practices manage to socially constitute mathematical domains?*

As a preliminary to answering this question, it will be useful to ask, "Why, according to social constructivists, is the purported construction that takes place within mathematics social in nature rather than individual in nature?" After all, it would appear that many mathematical domains are introduced by *individual* mathematicians rather than by groups of mathematicians. For example, it would appear that William Hamilton introduced the domain of quaternions and Georg Cantor introduced the domain of transfinite numbers.

In order to answer this question, we first need to be clear about what is meant by 'social in nature' rather than 'individual in nature'. I mean to be asking "Why are mathematical constructs *sharable*?" That is, why can both you and I—and any reasonably sophisticated human being—theorize about the *same* mathematical construct rather than each of us theorizing about, and thus constructing, a different mathematical domain. For example, it could be that you construct *your* domain of natural numbers and I construct *my* domain of natural numbers, where these two constructs are *different* entities.[17]

Consider for a moment another class of constitutive social constructs, sonatas. In general, one individual is responsible for composing any given sonata, yet this does not undermine the social—sharable—nature of sonatas. An individual's musical creation can be shared by many, because that individual uses socially recognized tools in its construction. For example, sonatas are composed using the twelve-tone scale, a social convention standardized around "middle C" having the frequency of 440Hz, and sonatas are composed for standard—socially recognized—musical instruments. It is precisely because shared musical tools of this type are used in the construction of sonatas that they are constructs of a social nature rather than constructs of an individual nature.

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[16] I shall provide a somewhat more detailed discussion of abstract entities in Section 5.

[17] There are some passages in Arend Heyting's work that suggest that he took mathematical entities to be individual mental entities rather than sharable entities in the sense that I am concerned with here (see, e.g., [Heyting 1931]). My worries about the sharability of mathematical constructs are a direct response to Gottlob Frege's criticisms of psychologism (see [Frege 1884]). My interest in this notion of "social", i.e., sharability, distinguishes me from Ernest and Hersh. Reflection on constitutive social construction will reveal that it is frequently achieved by providing certain individuals or groups of individuals with certain rights, responsibilities, authorities, etc. Consequently, it involves complex social dynamics. At least as I read Ernest and Hersh, when they claim that mathematical constructs are social in nature, they are acknowledging the importance of these social dynamics. I certainly do not want to deny the importance of these social dynamics. Likewise, I presume that Ernest and Hersh would not want to deny that mathematical constructs are sharable. We are merely emphasizing different things with our respective uses of the word 'social'.

According to social constructivists, a similar situation arises in mathematics. Frequently, one mathematician is responsible for the mathematical community taking an interest in a particular mathematical domain.[18] Consequently, from the perspective of a social constructivist, one individual is responsible for introducing the mathematical practice that constitutes that domain. Yet mathematical domains are sharable because mathematicians use shared logical tools (e.g., first and higher-order quantification) to characterize and constitute those domains.[19] These shared logical tools allow mathematicians to characterize the domains they seek to theorize about, specifically, how they are structured into objects, properties and relations. For example, characterizing the structure of the domain of natural numbers involves characterizing an ω-sequence.

Characterizing the structure of a mathematical domain is precisely what we take categorical axiom systems to do. For example, Hilbert's axioms characterize the structure of a two-dimensional Euclidean plane. The production of a categorical axiom system within a mathematical practice is, usually, the formal culmination of a long process. From the early stages of their development, mathematical practices that concern a single domain incorporate features that informally characterize the structure of the domain they concern. Perhaps the most important such features are the informal proofs and counterexamples given and accepted within the practice in question. Close consideration of which such proofs are judged legitimate, which illegitimate, and which purported counterexamples are taken to be actual and which not provides extensive information about the structure of the domain the mathematical practice in question is about. These and other features of the early development of these types of mathematical practices contribute to those practices determining how their subject matters are structured into objects, properties, and relations.

Let us now return to the question asked at the outset of this section, i.e., according to social constructivists, how, exactly, do mathematical practices manage to socially constitute mathematical domains? The optimal answer to this question—or at least part of that optimal answer—is that it is their ability to provide a (coherent) characterization of a particular structure.[20]

A social constructivist should maintain that all that there is to a particular mathematical domain existing is the undertaking of a mathematical practice that centers about a (coherent) characterization of the structure of the domain in question.[21] So, for example, when William

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[18] There are, of course, cases where two mathematicians are independently responsible for introducing a particular mathematical domain. I am not aware of any analogous cases in the musical world. This difference is best explained by the very specific purposes for which mathematical domains are introduced. Further, this difference in no way undermines the point I am making in this discussion.

[19] A second difference between the mathematical and musical cases relates to community involvement in the characterization of a construct. While occasionally close friends of a composer do make suggestions for change prior to the completion of a composition, typically, other musicians are extremely uncomfortable making any changes to another's (finalized) work. By contrast, it is common for members of the mathematical community to seek more conspicuous characterizations of newly introduced mathematical domains. Ernest and Hersh both emphasize this type of social negotiation as important to the nature of mathematics. I agree, but wish to note that this aspect of mathematics' social nature is independent of, and secondary to, the type of sociality that I am discussing. In order for a mathematician to offer a different characterization of a newly introduced domain, he or she must already be sharing the domain in question with the individual who introduced it. Thus, that domain must be sharable.

[20] There is, in fact, a lot more involved in providing an optimal answer to this question. Some further details can be found in Chapter 2 of my Ph. D. dissertation [Cole 2005].

[21] Early characterizations of new domains are often less than ideal. Frequently, later development shows them to be ambiguous. Difficult questions need to be asked about when the practices surrounding these characterizations actually

Hamilton first started to discuss entities with a noncommutative algebra to help represent and reason about 3-dimensional vectors, he introduced the practice responsible for the existence of quaternions.

With this social constructivist conception of mathematics in place, we should note the following features of it. First, it vindicates Hersh's claim that mathematical domains "detach from their originator" ([Hersh 1997], p. 16). They do this in a similar way to the way that a piece of music detaches from its composer or composers. Both types of detachment are made possible by the use of sharable tools in the social construction/constitution of the respective items.

Second, mathematical domains have objective features. Sonatas have objective features because of the objective features of the sharable tools that are used in their composition. Similarly, the objective nature of the logical tools used in the characterization and constitution of mathematical domains provides them with objective features. Roughly speaking, mathematical domains inherit the objectivity of logical consequence because they are constituted using logical tools.[22]

Third, the detachment of mathematical domains from mathematical practices allows for the epistemic independence of mathematical domains from mathematical practices highlighted in Section 2. Our imperfect knowledge of mathematical domains can be accounted for in the following way: mathematical domains are constituted using logical tools, yet human beings do not immediately perceive all of the logical consequences of a given characterization of a domain.

Fourth, the above social constructivist conception of mathematics at least points in the direction of an account of how finite mathematical practices have the ability to socially constitute mathematical domains with extremely large cardinalities. Mathematical practices do so simply by (coherently) characterizing those domains. There is, of course, an interesting question that one might ask about how mathematical practices manage to so characterize extremely large domains. Yet it is clear that mathematicians do take themselves to do this all of the time. Thus, *any* philosophy of mathematics will have to face this question concerning characterization (and offer an answer to it) unless it wants to claim that mathematical practices are riddled with massive amounts of error.

Fifth, according to the above account of the nature of mathematical domains, they are socially constituted by mathematical activities that concern particular mathematical domains (e.g., arithmetic, early Euclidean geometry, real analysis, complex analysis, and set theory). Those aspects of mathematics—such as group theory, ring theory, etc.—that do not concern particular domains (but rather all domains that share some structural features) do not, at least in general,[23] contribute to the social constitution of mathematical objects.

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become responsible for the existence of the domains they characterize. We need not address these difficult questions for our purposes in this chapter.

[22] Issues concerning the objectivity of logic and, consequently, the inherited objectivity of mathematics are complex from the perspective of a social constructivist. Unfortunately, I cannot hope to treat them adequately in this chapter.

[23] It seems to me that there are (probably) historical exceptions. Algebraic theories are only considered of interest if there are particular domains that have the structural features they center about. Consequently, a mathematician working on an algebraic theory will generally produce examples that have the structural features her theory centers about. Nowadays, set theory provides such examples. But, before this rich collection of structures was constituted by set-theorists, those working with algebraic theories produced their own examples. I suspect that occasionally this resulted in them characterizing new particular domains.

5 Abstract Entities

There is one final piece of metaphysics[24] that is worth exploring before we turn to the evaluation of social constructivism as an account of mathematical domains. This is the issue of whether or not mathematical domains and the items of which they are composed are abstract entities. Traditional—by which I mean Platonistic accounts of mathematical entities take them to be abstract entities—indeed, paradigm cases of abstract entities.[25] You might recall, however, that Hersh denies that mathematical entities (and social-historical entities in general) are abstract entities. Yet his argument for this thesis is peculiar. First, all he tells us about abstract entities is that they are neither mental nor physical. Second, he maintains that social-historical entities are neither mental nor physical. Why, then, does Hersh deny that social-historical entities are abstract? The reason, I suspect, is that Hersh's concept of an abstract entity is—unnecessarily—restricted to the concept of a paradigm case of an abstract entity.

A concept F is said to be a **cluster concept** if the application of F is determined by several features, known as "the cluster constitutive of F." If F is a cluster concept, then an item x is F if and only if x has a sufficiently large number of the features in the cluster constitutive of F. An item that has *all* of the features in the cluster constitutive of F is said to be a **paradigm case of F**.[26] I contend that 'abstract' is a cluster concept. It is difficult to specify all members of the cluster constitutive of '**abstract**', but the following are the most important members: **acausality**—the item neither exerts a (strict) causal influence over other items nor does any other item influence it in a (strict) causal way,[27] **non-spatio-temporality**—the item does not stand in spatio-temporal relations to other items, **eternality**—the item exists timelessly, and **changelessness**—none of the item's (intrinsic[28]) properties change. I conjecture that, for Hersh, for something to be an abstract entity, it must have all of these features and the others in the cluster constitutive of 'abstract'.

Hersh's restricted use of abstract is quite understandable and his claim that social-historical entities are not abstract is reasonable. Many social-historical entities fail to have some of the features constitutive of 'abstract'. For example, the U.S. Constitution has a causal impact on people, was constituted at a certain time and so is not eternal, and—perhaps[29]—goes through revisions of its intrinsic properties: amendments to it have been, and probably will continue to

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[24] **Metaphysics** is the branch of philosophy that investigates the nature of reality. A metaphysical account of some subject matter is a theory about the nature of that subject matter. The following theses are popular parts of *Platonistic* metaphysical accounts of mathematics: mathematical entities exist, mathematical entities would exist even if there weren't any human beings or other types of beings, mathematical entities are not spatio-temporal entities, mathematical entities do not causally influence other entities, the properties of mathematical entities do not change over time, etc. I hope that these theses give the reader some understanding of what it is to provide a metaphysical theory (or interpretation) of mathematics.

[25] I shall provide an account of what an abstract entity is and what a paradigm case of an abstract entity is shortly.

[26] This notion of a cluster concept is prefigured in a number of places in the philosophy literature. Perhaps the most useful discussion is Hilary Putnam's (see [Putnam 1962]).

[27] The relevant sense of strict is the one I identified while I was discussing causal social construction.

[28] The intrinsic properties of an item are those that it has independently of its relationships to other items. This modifier is needed, because it is clear that the extrinsic properties of all things change. For example, the extrinsic properties of the number 7 would change were I to decide that it is no longer my favorite (natural) number.

[29] There is a very tricky issue here about whether such amendments result in a new Constitution or a modified version of the original Constitution.

be, made. It is thus quite reasonable that Hersh should take mathematical entities to be like other social-historical entities in this regard.

However, I don't see any convincing reason why a social constructivist *has* to deny that mathematical domains and the items of which they are composed are acausal, non-spatio-temporal, eternal (or at least timeless), and changeless. I have sketched an argument elsewhere that this suggestion is intelligible (see [Cole 2005], Section 2.1). In fact, in [Cole 2005], I actually endorse it. Unfortunately, I do not have space here to provide a full argument for my endorsement of this suggestion. At the heart of this argument is a recognition of the universal representational function that mathematical domains serve. In essence, the argument is that the universal representational function of mathematics would be undermined by our taking mathematical domains to be causal, spatio-temporal, of limited duration, or changeable.

For clarity, let me briefly illustrate what I mean by the universal representational function of mathematics. The natural numbers can aid us in representing *all* subject matters—including past, future, spatio-temporal, abstract, and counterfactual subject matters. For example, I can claim that the number of people on planet Earth was smaller one hundred years ago than it is today and than it is likely to be in one hundred years time. Mathematics' ability to help represent all subject matters is what is meant by the claim that mathematics' representational function is universal.

A further reason a social constructivist should maintain that mathematical domains are abstract entities is the abundance of tenseless forms of representation in mathematical practices. Another is the fact that this contention allows for the vindication of the intuition that $2 + 2 = 4$ has always been true, as have all well-established mathematical truths.[30]

In addition, maintaining that mathematical domains and the items of which they are composed are (at least close to) paradigm cases of abstract entities would allow a social constructivist to sidestep some tricky issues. For example, it is well-known that Newton's and Leibniz's early developments of calculus were riddled with inconsistencies, yet practiced users of Newton's and Leibniz's tools were able to avoid these inconsistencies. Does the presence of this stable mathematical practice *force* a social constructivist to acknowledge the existence of a domain of infinitesimals with inconsistent properties constituted by this practice? On the present proposal, the answer is no. She *could*[31] take Newton and Leibniz to have been making a range of false assumptions about the real numbers as constituted by our contemporary practice of real analysis—presuming, of course, that our practice of real analysis does constitute the domain of real numbers. Further, the contention that mathematical domains and the items of which they are composed are (at least close to) paradigm cases of abstract entities would allow a social constructivist to account for mathematical practices progressing toward optimal characterizations of mathematical domains. It would also provide for a sense in which a social constructivist could account for early participants in a mathematical practice—individuals like Newton and Leibniz—getting things wrong about the domain the practice in question concerns. Both the claim that mathematical practices progress toward optimal characterizations of mathematical domains and the claim that

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[30] Some social constructivists (e.g. Ernest and Hersh) would deny or criticize this intuition. I do not share their views on this matter.

[31] She is not, however, forced to offer this answer. A careful investigation of the early practices surrounding the calculus might warrant her accepting the constitution of a domain having inconsistent properties.

early participants in mathematical practices get things wrong about the domain the practice in question concerns find widespread acceptance in our everyday thought about mathematics.[32]

It is for the types of reasons mentioned above that I take the optimal variety of social constructivism to be one that takes mathematical domains and the items of which they are composed to be constituted as (at least close to) paradigm cases of abstract objects. For convenience, let us call this variety of social constructivism **practice-dependent realism (PDR)**—"realism" because it maintains that many mathematical domains genuinely exist and have objective features, and "practice-dependent" because their existence is dependent on the existence of the mathematical practices that constitute them. Officially, practice-dependent realism—like Platonism—is the conjunction of three theses about mathematical domains: a) some exist, b) they and the items of which they are composed are (at least close to) paradigm cases of abstract objects, and c) they and the items of which they are composed are dependent on mathematical practices—in fact, they are pure constitutive social constructs constituted by mathematical practices.

6 Why Accept Practice-Dependent Realism?

So far, I have done little more than explicate social constructivism in general and PDR in particular. I have given little or no reason to accept PDR—or any variety of social constructivism—as an account of the nature of mathematical domains. All that I have done is show that PDR is compatible with various aspects of mathematical practices. So, why might one endorse PDR? Ernest pays little attention to this aspect of his social constructivist proposal. I find no argument in his book for social constructivism—the thesis that mathematical domains are socially constituted by mathematical practices. Hersh, on the other hand, provides two arguments in favor of social constructivism.

Hersh's first argument is an extended historical discussion of Platonism, social constructivism's best-known rival. This discussion shows why Platonism has been the historically dominant account of mathematical domains. It also demonstrates why the historical factors that have made Platonism dominant do not provide it with genuine support.

It is all very well to show that Platonism has been accepted for dubious reasons: despite this, it might be true. As Hersh notes in connection with mathematical discovery, it doesn't matter how you come to believe a thesis, what matters is whether that thesis is true. What is needed is not an argument that historical arguments for Platonism have been flawed, but an argument that Platonism is false or at least that social constructivism is preferable to Platonism as an account of mathematical domains. Even better would be a positive argument for the conclusion that social constructivism is true.

Hersh's second argument takes a very different approach from his first. It cites historical evidence from mathematical practices concerning the creative nature of mathematicians' activities. Specifically, Hersh argues that introducing new mathematical theories is a creative endeavor, i.e., involves genuine creativity. No doubt Hersh is correct; the introduction of new mathematical

[32] The arguments of this paragraph rely on the assumption that a mathematical discursive practice is able to pick out a mathematical domain as the one it is about, even if it does not characterize that domain perfectly. This is a controversial assumption. Yet my overall argument on behalf of social constructivism can be provided without the support of the arguments made in this paragraph.

theories is a creative endeavor. Yet this fact does not establish the truth of social constructivism. The introduction of new theories about the spatio-temporal world is also a creative endeavor. It doesn't follow that the spatio-temporal world is a pure constitutive social construct. So, Hersh's second argument is no more successful than his first.

Yet Hersh's second argument does point in the direction of a better argument for social constructivism, or at least the thesis that social constructivism is preferable to Platonism as an account of the nature of mathematical domains. In order to make this better argument, one needs to provide historical evidence for more than the thesis that the introduction of new mathematical theories is a creative endeavor. One needs to provide historical evidence that it is a creative endeavor that is not—in fact, cannot be—influenced by **Platonistically construed** mathematical domains—domains that are abstract and independent of all social practices. To start with, such evidence would distinguish the mathematical case from the spatio-temporal case; clearly, theories about the spatio-temporal world are generated under the influence of that (independent) world. In addition, however, it would go some way toward establishing that Platonistically construed mathematical domains are a kind of metaphysical extravagancy that we can—and therefore *should*—do without.[33]

In fact, Hersh even makes observations about mathematical practices that support the conclusion that new mathematical theories are introduced by mathematicians without influence from Platonistically construed mathematical domains. Recall, for example, the second half of Hersh's Fact 1, where he tells us that the creation of mathematical domains is "[not arbitrary], but from activity with existing mathematical objects, and from the needs of science and daily life" ([Hersh 1997], p. 16). Here Hersh observes that new mathematical theories are, on the whole, introduced for two reasons. Most (particularly contemporary) mathematical theories are introduced in response to needs internal to mathematics, such as answering questions raised within already existing mathematical practices. Consider, for example, Hamilton's introduction of the quaternions as a tool for representing and reasoning about three dimensional vectors. The other important reason why new mathematical theories are—or at least were—introduced is in response to a need from science or everyday life, frequently the kind of representational need mentioned in Section 5.

It is important to recognize that new mathematical theories are not introduced simply to describe Platonistically construed mathematical domains that, to use Gödel's famous phrase, "force themselves upon" us. Yes, once mathematical domains have been characterized by some individual or group of individuals, discovering their properties can—and does—feel like discovering the properties of something external to the individual.[34] But this kind of feeling comes after the introduction of a new mathematical theory; it does not motivate that introduction. The introduction of mathematical theories occurs for a variety of (other) reasons, primarily the two mentioned in the last paragraph. These reasons can force a mathematician to include certain features in her new theory, but these external constraints are not constraints from a Platonistically construed mathematical domain, but rather from the problem that she is introducing her theory to solve. Further, mathematical domains can—and, a social constructivist will argue, do—perform

[33] Further details of how this argument is meant to go will be provided in Section 8.

[34] Indeed, according to social constructivists, these properties are external to individual mathematicians in the sense that they are determined, at least to a large extent, by the objective logical tools used to characterize mathematical domains.

the roles demanded by these reasons without them needing to exist independently of mathematical practices.

In an ideal world, I would provide further empirical evidence showing that new mathematical theories *are* introduced without influence from Platonistically construed mathematical domains. But space is limited. So, instead, I shall offer a philosophical argument that new mathematical theories *must be* introduced without influence from these domains. In order to make this argument clear, it will be useful to relate it to well known epistemological worries about Platonism. So, let us consider those.

7 Platonism and Epistemology

In his 1973 paper "Mathematical Truth" [Benacerraf 1973], Paul Benacerraf made explicit an epistemological concern about Platonism that has inspired much discussion. It is now generally agreed that Benacerraf's original formulation of the challenge is not damaging to Platonism because it rests on a false assumption. Benacerraf's original formulation assumes that there needs to be a causal relationship between a knower and any domain of which she has knowledge. Yet the influence of his challenge remains, as it has been reformulated without mention of this false premise. Perhaps the most forceful such reformulation is Hartry Field's (see [Field 1989]).

According to Platonists, there are two distinct realms that are connected in a specific way: first, a mathematical realm consisting of Platonistically construed mathematical domains, and second, a collection of beliefs, shared by many mathematicians (and others), about this mathematical realm. Further, according to Platonists, the mathematical domains that make up the mathematical realm in question are those things that make the mathematical beliefs in question true or false.[35] Thus, the connection that Platonists claim holds between these two realms is that the first makes many of the second true. Given mathematicians' (and non-mathematicians') causal isolation from any Platonistically construed mathematical realm, there is a need for an explanation of this connection existing. Field challenges Platonists to provide such an explanation.

Let us call an explanation **non-mysterious** if it does not appeal to any mechanisms that would be found illegitimate by a reasonable individual engaged in a natural scientific investigation of the world. The specific form of Field's challenge to Platonists is to provide a non-mysterious, even if only rough, explanation of the *systematic* truth of mathematicians' (and non-mathematicians') pure mathematical beliefs. In other words, Field challenges Platonists to identify some collection of mechanisms that are scientifically investigable and which, in principle, could be the basis of an explanation of mathematicians (and others) having systematically true beliefs about a Platonistically construed mathematical realm.

Field's challenge is legitimate because we share a belief that non-mysterious explanations are, in principle, available for many types of relationships, including our knowledge and beliefs about the world. It is therefore unacceptable to provide an account of the nature of mathematical reality that rules out the possibility of there being a non-mysterious explanation of our having systematically true beliefs about that reality. Field challenges Platonists to show that they have not made this unacceptable move.

[35] More precisely, the mathematical realm is that in virtue of which the mathematical beliefs are true or false.

It is our lack of causal connection with any Platonistically construed mathematical realm that motivates Field's challenge. Yet Field's challenge is stronger than can be recognized simply by noting our lack of causal connection with such a realm. 'Abstract' is, in fact, defined in opposition to 'spatio-temporal'. Thus, the abstract nature of any Platonistically construed mathematical realm makes it likely that *all* explanations grounded in features of the spatio-temporal world are unavailable to a Platonist in answering Field's challenge. In other words, it is likely that there are no scientifically investigable mechanisms that could be the basis of an explanation of a Platonistically construed mathematical realm influencing mathematicians and their practices.

Logical deduction is likely to occur almost immediately to the reader as a different kind of potential tool for responding to Field's challenge. Yet noting the role of deduction in mathematics does not provide a full answer to Field's challenge, because beliefs established by means of deduction are only systematically true if the basic beliefs from which they are deduced are systematically true. So, for example, the many arithmetical truths that one can establish by deduction from the Peano Axioms are only systematically true if the Peano Axioms are. A Platonist must thus account for the systematic truth of the basic truths about mathematical domains. Most mathematicians will be tempted to suggest that the basic truths about mathematical domains are true in virtue of something like stipulation. But *why* can we simply stipulate these basic truths? The independence of Platonistically construed mathematical domains from mathematical practices seems to ensure that there might be no mathematical domain that answers to the stipulations in question.

A similar challenge has no force against a PDRist, because, according to her, mathematical *practices* are responsible for mathematicians' (and non-mathematicians') basic pure mathematical beliefs being true. The Peano Axioms are true because they have been accepted as an optimal characterization of a collection of objects, i.e., the natural numbers, appropriate for mathematicians' purposes. More generally, because mathematicians get to decide which mathematical objects should be constituted to serve their purposes, and get to decide which basic claims best characterize such objects, roughly speaking, mathematicians do indeed stipulate the basic truths about mathematical domains. Further, mathematical practices, as spatio-temporally instantiated activities, can causally influence human beings to become (at least minimally) competent participants in them. Consequently, mathematical practices can causally influence human beings to have systematically true pure mathematical beliefs. Think, for example, of how school teachers influence their pupils to become minimally competent participants in mathematical practices. This influence begins with such rudimentary lessons as how to add together two natural numbers, includes an introduction to axiomatic characterization and deduction from axioms, usually in the form of Euclidean Geometry, and will, in the mathematically sophisticated classroom, incorporate discussions of how to characterize the continuity of real valued functions using epsilons and deltas.

There are those who have not been persuaded by Field's challenge. While there have been many responses, the only promising one has been of the following type.[36] Field's challenge—and other challenges inspired by Benacerraf—rests on a false assumption: that there is some need for the mathematical realm to *influence* human beings in order for human beings to have mostly true beliefs about that realm. As Mark Balaguer (see [Balaguer 1998]) and Stewart

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[36] A defense of this claim can be mounted along the lines found in Chapter 2 of [Balaguer 1998].

Shapiro (see [Shapiro 1997]) have realized, such influence is not required. All that is needed is that no matter which axiom systems mathematicians choose to believe, provided only that they are coherent,[37] they will be true of that realm. Balaguer and Shapiro suggest that the mathematical realm is so "large" that every coherent axiom system (characterization) will be true of some collection of objects, properties, and relations in that realm. So, their response to Field is simple: *all* mathematical statements true in a coherent mathematical theory are true because the mathematical realm is so large that it has enough objects, properties, and relations to accommodate the existential commitments of the theory in question. This suggestion provides a partial[38] solution—indeed, the only known and ever likely to be produced partial solution—to the epistemological challenge to Platonists.[39]

For our purposes, it is important to note that this partial solution does not rest on any assumption that a Platonistically construed mathematical realm influences mathematicians or their practices. Indeed, according to both Balaguer and Shapiro, mathematical practices float free of any influence from the mathematical realm that they countenance. This mathematical realm is not involved in the best explanation of why mathematical practices are the way that they are. Nor should it be invoked as the basis of internal justifications of mathematicians' beliefs and choices.

Now, why have I gone to the trouble of discussing these epistemological worries about Platonism? First, one of the theses that is central to Field's challenge is that there is no respectable sense in which a Platonistically construed mathematical domain can be taken to influence mathematicians and their practices. My argument for the preferability of PDR to Platonism relies on this thesis. Second, some readers might be aware of the Balaguer-Shapiro response to the epistemological worries about Platonism. For this reason, I wanted to be clear that my argument is distinct from the argument to which Balaguer and Shapiro respond. Third, by giving the details of Field's challenge and the Shapiro-Balaguer response to it, I can make it clear that the Shapiro-Balaguer response does not undermine the thesis on which my argument relies.

8 Platonism vs. Practice-Dependent Realism

Let us turn to my argument for the preferability of PDR to Platonism as an account of mathematical domains. It is very simple: *Platonistically construed mathematical domains are explanatorily and justificationally superfluous. Consequently, we should not accept their existence.* Let me make some observations about this argument. First, the conclusion follows from the premise by means of an application of **Occam's razor**—don't multiply types of entities without necessity. The idea is that if Platonistically construed mathematical domains are explanatorily and justificationally superfluous, we can do without them.

[37] The notion of coherence in play here is a technical one developed by Shapiro (see [Shapiro 1997]). It is closely related to, though not identical with either, deductive consistency and set-theoretic satisfiability.

[38] I describe this solution as merely partial, because it leaves unjustified Platonists' metaphysical claims about the mathematical realm. A full solution to the epistemological worries about Platonism should have the resources to justify these claims.

[39] This claim is controversial. The most significant challenge to it concerns worries about our ability to refer to items in a realm consisting of Platonistically construed mathematical domains (see, e.g., [Azzouni 2000]). It falls outside the scope of this chapter to respond to this worry. Yet if this worry were to be well-founded, it would only strengthen the case against Platonism.

Those familiar with mathematical practices might be wary of applying Occam's razor to mathematical domains. Mathematics is not governed by Occam's razor. Rather, it is an underlying methodological feature of many mathematical practices that mathematicians should seek maximal generality. This feature of those practices can, particularly in foundational areas such as set theory and category theory, result in the characterization of ever larger mathematical domains. There is no problem here, however, because my application of Occam's razor is not internal to some mathematical practice, but rather takes place within the practice of naturalistic metaphysics (i.e., metaphysics guided by the methodological practices of natural science). Occam's razor is a legitimate tool within this practice, because it is a legitimate tool within the non-mathematical aspects of natural science.

Further, I take it to be a benefit of PDR that it predicts this methodological difference between the mathematical and non-mathematical aspects of natural science. If mathematical domains are pure constitutive social constructs, then Occam's razor governs mathematics if and only if it governs the practices that constitute pure constitutive social constructs. Does it? No! Consider for a moment the collection of legal statutes of the United States of America. Without doubt, the system of law embodied in this collection could be represented in a simpler and theoretically more elegant way by a collection of statutes with fewer members than there are in the actual collection. Despite this, we claim that the number of legal statutes is exactly the number in the actual collection. That number is, at least roughly speaking, the number felt necessary in order for them to serve the social functions for which they are constituted. So, the proposal that mathematical domains are pure constitutive social constructs should bring with it two predictions: first, that Occam's razor does not govern mathematical practices, and second, that the number of mathematical domains that in fact exist is linked with the purposes for which mathematical domains are constituted. Both predictions are accurate.

The discussion in the preceding paragraph points toward the following: entities that are *dependent* on social practices are not the kind of entities whose existence should be denied on the grounds of Occam's razor, while entities that are *independent* of social practices are the kind of entities whose existence can be—and, in some cases, should be—denied on the grounds of Occam's razor.

Let us now consider the premise of my argument, viz., Platonistically construed mathematical domains are explanatorily and justificationally superfluous. The thesis that there is no respectable sense in which these domains can be thought to influence mathematicians and their practices is central to the justification of this premise. Yet my premise requires further justification, for it might be possible for these domains to play some kind of explanatory or justificatory role without influencing mathematicians or their practices. Indeed, this belief has been embedded in a number of arguments for Platonism. For example, the existence of Platonistically construed mathematical domains has been argued to be required in order for mathematical statements to have the truth-value ascribed to them by mathematicians. Also, their existence has been considered necessary for providing mathematics with a semantics that resembles the semantics of everyday discourses sufficiently closely to account for the way in which these two types of discourses are intermingled.[40]

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[40] See [Benacerraf 1973] for arguments of both types.

Taking PDR seriously undermines both of these reasons for postulating Platonistically construed mathematical domains. First, PDR takes mathematical statements to have the truth-value ascribed to them by mathematicians. Second, since pure constitutive social constructs are among the entities talked about using everyday discourses, an adequate semantics for everyday discourses must be able to accommodate them.

Perhaps there are other explanatory or justificatory benefits that Platonistically construed mathematical domains might yield without influencing mathematicians or their practices. The most natural suggestion would be that they are indispensable to an account of the objectivity of mathematics. Yet—as I have indicated above—I believe that a PDRist has the resources to provide such an account. If such an account can be fleshed out, then Platonistically construed mathematical domains are not required for this purpose.

In fact, it is difficult to see what work Platonistically construed mathematical domains can do that the mathematical domains countenanced by PDR cannot do. And, unless some such work can be found—indeed, a fairly significant amount of such work can be found—we should not countenance these domains, for to do so would be to multiply types of (independent) entities without necessity.

9 Conclusion

Obviously, there is still much work that could be done in defending the premise of my argument that PDR is preferable to Platonism as an account of mathematical domains. Most importantly, I need to give the details of a PDRist's account of the objectivity of mathematics. There is also a need for arguments that PDR is preferable to the other accounts of mathematical domains found within the philosophy of mathematics literature. Such arguments require PDRists to show that they have the resources to account for the other traditional features of mathematics (e.g., its apriority and necessity). I don't have the space to explore these topics in this chapter. What I hope I have achieved in this chapter is to have given you a clearer understanding of what social constructivism about mathematics is and to have given you an idea of why you might want to be a social constructivist about mathematics.

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