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Abstract	In fall 2015 and spring 2016, I was lucky enough to be on sabbatical. My project was to complete a manuscript generalizing my institutional account of mathematics to cover other paradigmatically abstract objects. Ten years earlier, while finishing my dissertation, I had read Reuben's *What is Mathematics, Really?* At the time, I remember thinking I'm not really sure whether we agree about the nature of mathematical objects. On the one hand, Reuben's book was the first one that I encountered an explicit comparison between mathematical objects (e.g., numbers, circles, and ordered fields) and institutional objects (e.g., marriages, wars, corporations, and the US Supreme Court), a comparison that is central to my institutional account of mathematics. On the other hand, while Reuben seemed to be committed to there being mathematical and (other) institutional objects, many of the things that he said about them suggested that he viewed them more like fictions than genuine existents. I was intrigued, but didn't really follow up on that intrigue until my sabbatical offered me the opportunity to visit some of the folks who I took to endorse similar accounts of mathematics to my own. Two such individuals were Reuben and Sol Feferman. For me, our meetings revealed something interesting: while we were all three natural allies in taking humans to be responsible for mathematics—we were *humanistist*, to use Reuben's terminology—we did not agree on the underlying nature of mathematical objects. At least as I interpreted them, Reuben and Sol held fairly similar views about the underlying nature of such objects: they are something like intersubjective mental objects. Sol—see (Feferman 2009, 2014)—expressed his view in this way, "the basic objects of mathematical thought exist only as mental conceptions," where, according to Sol, these mental conceptions are highly constrained by social interactions concerning them. Reuben (2014, p. 13), on the other hand, expresses his view in this way: "A mathematical entity is a concept, a shared thought" and "The concept … is nothing other than the collection of the mutually congruent … 'mental models' … possessed by those participating in the mathematical culture." To summarize	

Reuben's view as he did in personal communications that followed our November 2015 meetings, "mathematical objects are 'equivalence classes' of mutually congruent 'mental models' of those objects."

Humanism About Abstract Objects

Julian Cole

Introduction

In fall 2015 and spring 2016, I was lucky enough to be on sabbatical. My project was to complete a manuscript generalizing my institutional account of mathematics to cover other paradigmatically abstract objects. Ten years earlier, while finishing my dissertation, I had read Reuben's *What is Mathematics, Really?* At the time, I remember thinking I'm not really sure whether we agree about the nature of mathematical objects. On the one hand, Reuben's book was the first one that I encountered an explicit comparison between mathematical objects (e.g., numbers, circles,[1] and ordered fields) and institutional objects (e.g., marriages, wars, corporations, and the US Supreme Court), a comparison that is central to my institutional account of mathematics. On the other hand, while Reuben seemed to be committed to there being mathematical and (other) institutional objects, many of the things that he said about them suggested that he viewed them more like fictions than genuine existents. I was intrigued, but didn't really follow up on that intrigue until my sabbatical offered me the opportunity to visit some of the folks who I took to endorse similar accounts of mathematics to my own. Two such individuals were Reuben and Sol Feferman. For me, our meetings revealed something interesting: while we were all three natural allies in taking humans to be responsible for mathematics—we were *humanistist*, to use Reuben's terminology—we did not agree on the underlying nature of mathematical objects. At least as I interpreted them, Reuben and Sol held fairly similar views about the underlying nature of such

[1] I am referring to perfect circles rather than the roughly circular objects that we find around us.

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B. Sriraman (ed.), *Humanizing Mathematics and its Philosophy*,
DOI 10.1007/978-3-319-61231-7_14

objects: they are something like intersubjective mental objects. Sol—see (Feferman 2009, 2014)—expressed his view in this way, “the basic objects of mathematical thought exist only as mental conceptions,” where, according to Sol, these mental conceptions are highly constrained by social interactions concerning them. Reuben (2014, p. 13), on the other hand, expresses his view in this way: “A mathematical entity is a concept, a shared thought” and “The concept … is nothing other than the collection of the mutually congruent … ‘mental models’ … possessed by those participating in the mathematical culture.” To summarize Reuben’s view as he did in personal communications that followed our November 2015 meetings, “mathematical objects are ‘equivalence classes’ of mutually congruent ‘mental models’ of those objects.”

I could not agree. On neither view, it seems to me, are there mathematical objects. On both views, we have only shared, or at least shareable, propositional, and nonpropositional mental representations of mathematical objects, where, to my mind, it doesn’t really matter what we call these representations. We do, of course, possess shared propositional and nonpropositional mental representations of mathematical objects of various kinds. For instance, all of us believe that “$2 + 2 = 4$,” many of us are aware of the conjecture that “every even natural number greater than 2 is equal to the sum of two prime numbers,” anyone who is reading this article will possess nonpropositional mental representations of the natural numbers and circles, and anyone with any mathematical sophistication will possess such representations of ordered fields. Yet these mental representations of mathematical objects are not mathematical objects, for we must keep separate mental representations of Xs, even shared mental representations of Xs from Xs.[2] Mental representations of chairs aren’t chairs, mental representations of tectonic plates aren’t tectonic plates, mental representations of unicorns aren’t unicorns, and mental representations of numbers, circles, and ordered fields aren’t numbers, circles, or ordered fields. For one thing, mental representations aren’t prime, circular, or ordered in the relevant sense. Sol, I am confident, would have agreed. But Reuben, I believe, does not; when I quoted him above as writing “A mathematical entity is a concept, a shared thought,” I did not quote a slip of the keyboard, so to speak, but a view of mathematical objects to which I believe he is committed. While Rueben recognizes the “logical” problem that I just identified, i.e., that mental representations of mathematical objects possess very different features than do mathematical objects, to date, at least, this problem has not convinced him to change his account of mathematical objects.

Moreover, Reuben’s account of mathematical objects has some important implications. For instance, mental models of mathematical objects of particular kinds are both dated and contingent. In particular, there is a time at which the first person to ever possess such a model comes to possess that model and a time at which any particular person first comes to possess such a model. Similarly, it is a contingent truth that there is a person who possesses a mental model of mathematical objects of a particular kind and that any particular person possesses such a model.

[2] There is, of course, one exception: when the Xs are, themselves, mental representations.

Furthermore, both our individual and collective mental models of mathematical objects of particular kinds are in a state of change; as we both individually and collectively investigate mathematical objects of particular kinds, our mental models of those objects change. Accordingly, if Reuben's account of mathematical objects is correct, three theses that philosophers of mathematics have spent significant effort explaining and justifying are false. Specifically, mathematical objects are neither atemporal existents nor necessary existents, nor unchanging; on the contrary, they are finite, contingent, and changing.

Reuben and Sol are not alone in holding views like those that I ascribed to them above (e.g., it is clear that George Lakoff and Raphael Núñez (2000) take the work of explaining "where mathematics comes from" to be complete once they have explained the origins of our mathematical concepts). Such views arise from the thought that, in order to explain mathematical practice and the role that mathematics plays in other activities (e.g., the development and statement of scientific theories), one really doesn't need to postulate anything other than (propositional and nonpropositional) mental representations of mathematical objects of particular kinds; in particular, one doesn't need to postulate mathematical objects themselves. The reasoning behind this thought goes something like this: given that mathematical objects, if there are any, are abstract, mathematicians, scientists, and others who take themselves to be investigating such objects or employing such objects in understanding reality really can't be investigating or employing these objects themselves. What then are they investigating and employing in their activities? They are investigating and employing shared, or at least sharable, mental representations of mathematical objects. Accordingly, what we really need to postulate to account for mathematical practice and the role that mathematics plays in other activities is not mathematical objects but shared mental representations of mathematical objects. And, if this is all that we need to postulate in order to do all of the explanatory work that is of interest to us, why postulate anything other than this? Why postulate "spooky," "mysterious," "otherworldly" objects that aren't facets of the spatiotemporal reality to which we are all, quite correctly, committed?

The answer to this question that many philosophers of mathematics accept is typically ascribed to Frege (1884). Here is a conceptual truth about the natural numbers:

The number of *Fs* is *n* if and only if there are exactly ***n*** *F*s.

Accordingly, provided that claims of the form "there are exactly ***n*** *F*s" are true (e.g., "there are exactly **four** computers in this room" and "there are exactly **two** objects that are identical to the natural number zero or the natural number one"), then so are claims of the form "the number of *Fs* is *n*" (e.g., "the number of computers in this room is four" and "the number of objects that are identical to the natural number zero or the natural number one is two"). On any reasonable analysis of the logical form and content of claims of the form "the number of *Fs* is *n*," the numerals that appear in these claims are singular terms that purport to refer to natural numbers. Moreover, it is standardly believed that claims of this particular logical form can be true only if the singular terms, i.e., numerals, that appear in them do in fact refer. Yet, if numerals that purport to refer to natural numbers do in fact refer, there are particular natural numbers. Indeed, if one uses the trick of considering "the objects that are non-self-identical," "the objects that are identical to the natural number zero," "the objects that are identical to the natural number one or the natural number two," etc., one quickly sees that all natural numbers exist.

So, there is a kind of explanatory work that requires us to acknowledge that there are natural numbers (and, by parallel reasoning, that there are mathematical objects of many other kinds): we need to provide a plausible and uniform semantics for our everyday claims. The problem with this observation is that it really doesn't allay the naturalistic/physicalistic concerns of those who, like Reuben, find the idea of acknowledging that there are mathematical objects that are distinct from our mental representations of such objects intellectually abhorrent since to do so, they believe, is to commit themselves to a "spooky," "mysterious," "otherworldly" platonism about such objects. That is, it is to commit themselves to there being mathematical objects that exist independently of our representations of them in a reality that is both distinct and causally/metaphysically isolated from the spatiotemporal reality that we occupy. How, they ask, could we ever come to know about such objects? How could we ever come to refer to them? Of what use could such objects be?

It is precisely at this point that the genius of Reuben's (1997) comparison between mathematical and institutional objects can be appreciated, for it points toward a way out of this dilemma. The US Supreme Court is not identical to our (propositional and nonpropositional) mental representations of it. It has the authority to interpret the law of the land and decide on matters of life and death; our (shared) mental representations of it do not have this authority. This court also isn't an everyday physical object in the way in which the computers in this room are; I can't literally pick it up and move it around. This court exists; it really does decide matters of life and death. Moreover, this court wouldn't exist if there weren't any human beings; in a straightforward modal-existential sense, it has an existence that depends on the existence of human beings. Furthermore, this court isn't "mysterious," "spooky," or "otherworldly" either; it exists because we represent it to exist and, give or take, it possesses the features that we provide it by representing it as possessing those features. The key point for our current purposes, though, is that the US Supreme Court and similar institutional objects clearly demonstrate that it is possible for humans to be responsible for there being objects that aren't everyday physical objects, aren't identical to our mental representations of them, and aren't "mysterious," "spooky," or "otherworldly" either. Accordingly, there shouldn't be a problem with maintaining that humans are responsible for there being mathematical objects that aren't everyday physical objects, aren't identical to our mental representations of them, and aren't "mysterious," "spooky," or "otherworldly" either. In other words, Reuben's comparison suggests that we can acknowledge the force of Frege's argument without being committed to a "spooky," "mysterious," "otherworldly" platonism about mathematical objects.

The devil, of course, is in the details. We need to know how, precisely, humans can be responsible for there being objects that aren't everyday physical objects, aren't identical to our mental representations of them, and aren't "mysterious," "spooky," or "otherworldly." I don't have the space to provide all of those details here, but let me at least outline some of them. First, note that, sometimes, we make reality a certain way by representing it to be that way. For instance, when, at the beginning of one of my classes, I utter "let's get started," I make it the case that my class has started by representing it as having started. Similarly, when a judge, in an appropriate setting, utters "I hereby order you to spend thirty days in prison," he or she thereby makes it the case that the person who he or she is addressing must spend thirty days in prison by representing him or her as having to do so. Formally, philosophers call these types of speech acts declarations, as in the (adoption of the) Declaration of Independence, which made it the case that there was a new nation by representing there to be such a nation. Informally, we may think of these actions as making it the case that things are certain ways or that there are certain objects by collective agreement.

Once one starts considering making things the case by collective agreement, one quickly realizes that many things are the case and that there are many objects by collective agreement. For instance, that I am an Associate Professor within the State University of New York (SUNY) system and that I am a British citizen are the case by collective agreement. Similarly, that there is a SUNY system, that there is the United Kingdom of Great Britain and Northern Ireland, and that there is the US Supreme Court are the case by collective agreement. Thus, if we take Reuben's comparison between mathematical and institutional objects seriously, the proposal to which we should be drawn is this: mathematical objects of particular kinds (e.g., natural numbers, circles, and ordered fields) exist by collective agreement, which, a little more formally, amounts to it being the case that mathematical objects exist by declaration, i.e., in virtue of us representing there to be such objects. I like to talk about objects that exist in virtue of us representing there to be such objects as having an existence that is *representationally dependent* on, i.e., *dependent*$_R$ on, collective intentionality.

Now, if one considers the collective agreements responsible for people possessing the statuses "associate professor within SUNY" and "British citizen" and for there being SUNY, the UK, and the US Supreme Court, it is clear that these agreements are relatively formal and legalistic. But there is no need for the collective agreements that are responsible for people or objects possessing particular statuses or for there being objects of particular kinds to be formal and legalistic. For instance, while I was younger, one of my friends would regularly make himself the goalkeeper in our makeshift game of soccer simply by standing in front of the makeshift goal on our makeshift field. Similarly, it is easy for me to make one of the two identical beers that I just bought at the bar David's and the other Juan's simply by handing them, respectively, to David and Juan. Likewise, someone can easily become a member of a particular clique simply in virtue of the current members of that clique allowing him or her to join them in their favored activities. In none of these cases are the relevant collective agreements either formal or legalistic.

Moreover, it isn't merely that people or objects can possess new statuses in virtue of informal and non-legalistic collective agreements; there can be objects in virtue of such agreements as well. For instance, in making a particular beer David's, I might make it the case that there is such an object as the 1,000th beer owned by

David, while, in surreptitiously joining a new clique, David might make in the case that there is such an object as the 10th member of the said clique. For our current purposes, though, the important point is that taking Reuben's comparison seriously doesn't demand that there be formal or legalistic collective agreements that are responsible for there being mathematical objects of particular kinds; one might be party to the relevant agreement(s) without ever explicitly formulating them or considering them, just as most of us are party to the agreement that one can make a beer some particular person's beer simply by giving it to him or her without ever explicitly formulating or considering this agreement.

Next, observe that we collectively agree that certain people and objects possess certain social statuses and collectively agree that there are objects of certain kinds in order for them to serve *functions*, i.e., in order for those people and objects to occupy roles that promote certain ends, goals, or purposes. For instance, collectively agreeing that there is such an object as SUNY facilitates the provision and certification of the higher education of numerous people, while collectively agreeing that I am an Associate Professor within the SUNY system facilitates me in performing my role in providing and certifying the higher education of students who attend a particular branch of this system. Accordingly, if we are to take Reuben's comparison seriously, we should be able to point to a function or functions that mathematical objects of particular kinds serve.

In service of identifying the function(s) that mathematical (and many other abstract) objects serve, let me make some observations. First, particularly when facets of reality are important to us, there are a variety of activities relating to them in which we frequently engage. These include reasoning about them, inquiring after their features, discovering truths concerning them, analyzing situations that involve them, planning actions that surround our interactions with them, and stating or representing how reality is with respect to them. Henceforth, label these and similar activities *representational* in recognition of the fact that they employ representations of the said facets of reality.

Second, we find it significantly easier to engage in representational activities when their *subject matter*—what we are inquiring after, discovering truths concerning, reasoning about, analyzing, representing, etc.—is a plurality[3] of objects or is treated as such in our representations. Indeed, we find engaging in such activities easiest when their subject matter is, or is treated as, a *small* plurality of objects, where this amounts to the said plurality containing four or fewer objects. That we find it significantly easier to engage in representational activities when their subject matter is, or is treated as, a (small) plurality of objects fits well with our everyday experience. For instance, anyone who has compared the difficulty of assessing the validity of a modal inference using representations of its premises and conclusion that employ possible worlds—a plurality of objects—and representations that do not will recognize how much easier such an assessment is when undertaken using

[3]For readability, I assume throughout that pluralities can be of any finite cardinality, including **zero**.

representations of the former type. Similarly, anyone with experience of doing things both ways will recognize how much easier it is to investigate the relationship between the cardinalities of various finite pluralities when one represents such cardinalities using natural numbers—a plurality of objects—than when one simply takes them to be features of finite pluralities.[4] Ultimately, however, the fact that we find it significantly easier to engage in representational activities when their subject matter is, or is treated as, a (small) plurality of objects is best accounted for by our innate representational capacities. Specifically, this fact is best accounted for by the cognitively basic nature of our capacity to represent reality in terms of (small) pluralities of objects, for this capacity ensures that many of the cognitive abilities and capacities that we possess for engaging in representational activities are configured to function best when we employ representations of reality that represent it in terms of (small) pluralities of objects.[5]

Third, we find it so much easier to engage in representational activities when their subject matter is a (small) plurality of objects that, when the subject matter of some representational activity R_N is not of this type, we often obtain R_N's outcome by using an alternate or *surrogate* representational activity R_S whose subject matter is a (small) plurality of objects. Or, to put this point as I shall throughout, we engage in R_N using a *surrogate subject matter* that consists of a (small) plurality of objects, i.e., we engage in R_N by treating its subject matter as a (small) plurality of objects by replacing its (non-surrogate) subject matter with a surrogate subject matter that is a (small) plurality of objects.[6] For instance, when we want to assess the validity of a modal inference, instead of doing so directly, we frequently restate its premises and conclusion using possible worlds and assess the restated argument rather than the original. Similarly, when we want to investigate the relationship between the cardinalities of certain finite pluralities, we frequently represent the said cardinalities using natural numbers and conduct the investigation in these restated terms rather than the original. Another way of expressing these points is that we employ possible worlds as a surrogate subject matter for the non-surrogate subject matter of various modal inferences and the natural numbers as a surrogate subject matter for the non-surrogate subject matter of various investigations into the relationships between the cardinalities of finite pluralities. Yet another way that I sometimes express points such as the aforementioned is that, when assessing modal inferences, we frequently employ possible worlds as surrogates for how reality would be if it were to be different in certain respects and for how reality could be for all that we know, while, when investigating relationships between the cardinalities of finite pluralities, we frequently employ natural numbers as surrogates for such cardinalities.

[4]For a convincing illustration of this second point, see (Field 1980, §2).

[5]Unfortunately, I do not have space in this article to explore and defend the cognitively basic nature of our capacity to represent reality in terms of (small) pluralities of objects.

[6]Here, I presuppose that we may individuate representational activities on the basis of their outcomes.

In illustrating my third observation, there is no need to consider only philosophically contentious objects like possible worlds and natural numbers, though. Consider, for instance, our division of continents into nations and towns/cities into plots of land. We undertake these divisions so that particular individuals and pluralities of individuals can claim ownership of the said nations/plots, where such ownership comes with special *deontic powers*—rights, responsibilities, authorizations, etc.—concerning the land in question. As such, it is a consequence of how we divide land for the aforementioned purposes that, at certain locations, there are transitions in deontology between individuals and/or pluralities of individuals. Moreover, given the importance that we place on ownership of land, the exact location of these transitions is frequently of great interest to us. When we engage in representational activities concerning these transitions, we almost invariably do so in terms of boundaries (aka borders) rather than the transitions themselves, i.e., we use a plurality of objects—boundaries—as a surrogate subject matter for these activities.

Next, consider complex organizations, such as large corporations, university systems, and intricate systems of government. Such organizations are associated with numerous people who, and pluralities of people that, contribute to their operation, and it is often difficult to understand the respective roles of these people and pluralities in the operation of their respective organizations. In order to facilitate such understanding for the purposes of engaging in various representational activities, it is common to represent organizations as possessing operational structures, where the positions in these structures serve as surrogates for people who, and pluralities of people that, perform particular roles in the operation of the said organizations. For instance, in explaining the US system of government, it is common to describe it as possessing three branches, which, in turn, contain such positions as those of the President, the Supreme Court, and the Congress. Moreover, that we collectively represent the operational structure of the US system of government as containing three branches that in turn contain small numbers of positions is no accident but illustrates our preference for engaging in representational activities using representations of reality that represent it in terms of *small* pluralities of objects rather than merely pluralities of objects of any cardinality. Our representations of the operational structures of complex organizations often take them to include layers that consist of small pluralities of positions since doing so allows us to engage in representational activities concerning these structures whose subject matters are small pluralities of objects.

As a third example, consider the game of chess. In teaching people to become better chess players or proving results about chess such as that it is impossible to force a checkmate against a lone king with a king and two knights, we frequently represent games of chess as mere sequences of moves, i.e., we engage in representational activities concerning the (token-individuated) games of chess in which actual players participate using the objects that philosophers call type-individuated games of chess as a surrogate subject matter for the non-surrogate subject matter of these activities.

It should now be clear that serving as a surrogate subject matter for a particular representational activity or for representational activities of a particular kind is a function that objects of a particular kind can usefully serve. This is the function that, in developing Reuben's comparison between mathematical and institutional objects, I take mathematical objects to serve; for convenience, I talk about mathematical (and numerous other abstract) objects as serving *surrogacy functions* in representational activities.

Given the immense variety of (non-surrogate) subject matters that aren't (small) pluralities of objects with respect to which we might wish to engage in representational activities, it would be nothing short of miraculous if there were pluralities of objects that exist independently$_{R}$ of intentionality that could serve as surrogate subject matters for all of these activities. More plausibly, we make it the case by declaration/collective agreement that there are pluralities of objects that can serve the relevant surrogacy functions; that is, we make it the case by declaration/collective agreement that there are mathematical (and many other abstract) objects.

For clarity, let me further explain the basic idea behind my proposal that we make it the case by declaration/collective agreement that there are mathematical objects of particular kinds. As I see things, during the course of various undertakings, mathematicians come to have an interest in investigating the consequences of facets of reality standing in certain relations to one another, which can be thought of them coming to have an interest in investigating the consequences of a plurality of facets of reality possessing a particular structure, where this investigation can also be understood as an investigation into the features shared by all possible pluralities of facets of reality that stand in the said relations to one another/possess the relevant structure. This investigation can more easily be carried out using a surrogate subject matter that consists of a plurality of objects that stand in the relevant relations to one another/possess the relevant structure as a surrogate subject matter, where, give or take, these objects possess only features that are consequences of them standing in the said relations to one another/possessing the said structure. In other words, this investigation can be carried out more easily using a particular mathematical structure as a surrogate subject matter, where a *mathematical structure* is a special type of object that is made up or constituted by other objects—its *positions* or *places*—that stand in certain particular relations to one another—in this case, the relations that are of interest—and are metaphysically incomplete in that, give or take, the only features that the places of a mathematical structure possess are those that are consequences of them standing in the particular relations to one another that they do. Given this, mathematicians make it the case by declaration/collective agreement that there is such a mathematical structure in order for it to serve as the surrogate subject matter of this particular investigation, though it should be remembered that this happens in an informal and non-legalistic way rather than by means of some formal, legalistic declaration/collective agreement. Investigating the consequences of a plurality of facets of reality possessing a particular structure by investigating the features of a mathematical structure whose places possess that structure is just how mathematicians go on in most situations, much as making a beer a particular person's by giving it to him or her is just how most of us go on in most situations. In each case, there is a background institution, with suitable constitutive rules/standing declarations, that makes our doing so perfectly appropriate.[7] In the latter case, the relevant institution is property ownership, particularly some of the more informal elements of this institution, while in the former case, it is what I have elsewhere called the *surrogate subject matter institution*. The central constitutive rule/standing declaration of this institution can be expressed in this way: our possessing a concept of a plurality of objects with particular features that, were it to be used as a surrogate subject matter for a given representational activity, would facilitate our engagement in that activity suffices for there being that plurality of objects/subject matter.

Perhaps a more concrete, historical example will aid the reader in understanding my account of mathematical objects/structures. While investigating algebraic solutions to cubic and quartic equations, sixteenth century mathematicians recognized that the domain of the square root function should include negative numbers, which, in turn, led them to recognize that there are possible outputs of this function that cannot be identified with real numbers, which, in turn, allowed them to form a shared concept of the complex number structure, i.e., a mathematical structure whose places stand to one another in the relations that *all* outputs of the square root function do. These mathematicians decided to investigate the consequences of facets of reality standing to one another in these particular relations. The central constitutive rule/standing declaration of the surrogate subject matter institution was, and continues to be, responsible for there being the complex number structure to serve as a surrogate subject matter for this investigation.

In understanding the surrogate subject matter institution and its central constitutive rule/standing declaration, it is helpful to compare it with other institutions that are responsible for there being objects of particular kinds. One such institution is corporate activity in the State of California, which is governed by the constitutive rules outlined in the State of California's Corporations Codes. Section 200a of these Codes reads "One or more natural persons, partnerships, associations or corporations, domestic or foreign, may form a corporation under this division by executing and filing articles of incorporation." This is a standing declaration/collective agreement to the effect that the right kind of individual(s)—one or more natural persons, partnerships, associations, or corporations, domestic or foreign—may make it the case that there is an object of a particular kind, a Californian corporation, by performing the right kind of actions, executing and filing

[7] As I understand an *institution*, it is a plurality of activities governed by constitutive rules, which are *standing declarations*, i.e., declarations that are in place for an extended period of time and, during that time, specify that fulfillment of certain conditions suffices for something being the case. For instance, basketball is an institution, and one of its constitutive rules/standing declarations specifies that throwing the ball through the opposing team's basket from outside the three-point line during play suffices for your team scoring three points. Please note that my use of institution is somewhat different from a colloquial one according to which institutions are what I earlier called organizations (e.g., universities and corporations).

articles of incorporation. Of course, the reality is somewhat more complicated than this in that the State of California has to recognize the execution and filing of the articles in question as legitimate, etc. Yet, give or take, the aforementioned is all that is involved in making it the case that there is a new Californian corporation.

There are, of course, some differences between how the institution of corporate activity in the State of California is responsible for there being Californian corporations and how the surrogate subject matter institution is responsible for there being mathematical (and other abstract) objects. For instance, the constitutive rules/collective agreements governing the former institution are highly formal and legalistic, while those governing the latter are informal and non-legalistic, Californian corporations serve very different functions than surrogate objects of various kinds—roughly, limiting the financial responsibilities of various people who are associated with them—and all that needs to happen for there to be surrogate objects of a particular kind is that someone formulate a shareable concept of those objects, while certain kinds of individuals must execute and file articles of incorporation for there to be a particular Californian corporation. But these are differences of degree, not differences of kind; essentially, the two institutions work in the same way to make it the case that there are objects of particular kinds.

So, drawing on Reuben's comparison between mathematical and institutional objects, I have been able to provide an account of how humans can be responsible for there being mathematical (and other abstract) objects: institutions, which are simply activities governed by clusters of standing declarations/collective agreements, can be responsible for their existence. What of the features of such objects? One claim that I have made repeatedly is that mathematical objects are abstract; how can this be the case without our being committed to "spooky," "mysterious," "otherworldly" objects?

To begin, let us consider what it means for an object to be abstract. I favor a strategy for specifying what abstract objects are that has come to be known as "the way of negation," that is, I take abstract objects to be those that lack the features characteristic of being concrete. Moreover, I take both "abstract" and "concrete" to be governed by a plurality of features rather than a single feature, where some of the features in these pluralities are more important than others to whether an object is, respectively, abstract or concrete. More specifically, an object is *abstract* if and only if it fails to possess *the most central feature* in the plurality associated with concrete, while the more such central features it fails to possess, the more paradigmatically abstract it is.

In order of centrality, the following are, according to my account, the most central features of the relevant pluralities:

	Concrete	Abstract
1.	Spatial/spatiotemporal	Nonspatial/non-spatiotemporal
2.	Causally efficacious	Causally inefficacious/acausal
3.	Exists for a finite period	Eternal/semi-eternal/atemporal
4.	Contingent existent	Necessary existent

As noted above, I take only the first of these features to be essential to being, respectively, concrete or abstract. Moreover, since we developed the concepts “concrete” and “abstract” before we understood that space and time are linked, the features listed under 1 can be understood in purely spatial rather than spatiotemporal terms. Accordingly, what I take to be essential to being abstract is lacking a spatial location; having any of features 2 through 4 in addition simply makes an object more paradigmatically abstract.

Now, what are the implications of this account for our making it the case by declaration/collective agreement that there are abstract objects? Put simply, they are that we can make it the case by declaration/collective agreement that there are abstract objects simply by collectively agreeing that there are objects that fail to possess a spatial location, while we can make it the case by declaration/collective agreement that there are paradigmatically abstract objects simply by collectively agreeing that there are objects that not only fail to possess a spatial location but also fail to causally interact with other objects, fail to be finite existents, and fail to be contingent existents. Admittedly, it can be difficult to understand how we might achieve the last of these things and, perhaps, the penultimate one as well. But it isn’t difficult to understand how we might make it the case that objects of a particular kind that exist by collective agreement fail to possess a spatial location or fail to causally interact with other objects: we just adopt a collective agreement that fails to provide the said objects with a spatial location or to specify any causal relations in which they stand. In other words, there is nothing “spooky,” “mysterious,” or “otherworldly” about our collectively agreeing that there are mathematical (and other abstract) objects; all that is involved in our doing so is our adopting of collective agreements that fail to provide the objects for which they are responsible with spatial locations and a causal profile.

We can also make sense of the idea that declarations/collective agreements are responsible for there being objects that are atemporal and necessary existents, but to do so, we must first consider some background issues. Most importantly, we need to understand what it would be for objects of a particular kind to exist atemporally or exist necessarily. Moreover, in order to understand this, we need to understand something more fundamental: what we are claiming in claiming that there are objects of a particular kind. Frege (1884) offered a simple account: there are objects of kind K if and only if the extension of the concept “K” is nonempty. I want to offer a slightly modified version of this account. In this connection, observe that sortal concepts and nominative terms that are associated with sortal concepts come with two types of conditions: *application conditions* and *coapplication conditions*. The former are conditions that must be met in order for it to be permissible to apply the relevant term or concept in some representational activity, while the latter are conditions that must be met in order for a permissible reapplication of a term or concept in a representational activity to be a coapplication of that term or concept, i.e., an application of it to the same object of the relevant kind. With these definitions in place, it is clear that there are objects of kind K if and only if it is permissible to apply the concept “K” in representing reality (as it actually is now), i.e., if and only if the application conditions for “K” are met by reality (as it actually is now).

Before I extend this understanding of existence to include what it is for objects of kind K to be necessary and/or atemporal existents, let me make an observation. We engage in representational activities not only concerning how reality actually is now but, for instance, concerning how reality would be were it different in certain respects, i.e., under subjunctive suppositions, concerning how reality might be for all that we know, i.e., under indicative suppositions, concerning how reality is according to a particular fiction, i.e., under fictional suppositions, concerning various times that are not now and even concerning no time at all. With this observation in place, the basic ideas behind what it is for objects of kind K to be necessary and/or atemporal existents may be communicated in this way. Objects of kind K are *necessary existents* if and only if the concept "K" possesses *subjunctively universal permissible coapplicability*, i.e., for any application *A* of "K," there is no subjunctive supposition *S* such that the mere fact that a representational activity *R* takes place under *S* makes it impermissible to coapply *A* in *R*. Similarly, objects of kind K are *atemporal existents* if and only if the concept "K" possesses *temporally universal permissible coapplicability*, i.e., for any application *A* of "K," there is no time *T* such that the mere fact that a representational activity *R* concerns *T* makes it impermissible to coapply *A* in *R*. Yet, while these are the basic ideas behind what it is for objects of kind K to be necessary and/or atemporal existents, usually, when we claim that objects of kind K are necessary and atemporal existents, we actually convey more than the aforementioned; usually, we convey that "K" possesses *universal coapplicability*, i.e., for any application *A* of "K," there are no restrictions on the permissible coapplication of *A*. In other words, in claiming that objects of kind K are necessary and atemporal existents, we are usually conveying that, in *all* representational activities, regardless of whether they take place under certain suppositions, concern times that are not now, or concern no time at all, the conditions for a permissible coapplication of any application of "K" are met.

With this understanding of what we are conveying in claiming that objects of kind K are necessary and atemporal existents, it is relatively easy to see how it could be that some surrogate objects are necessary and atemporal existents even though declarations/collective agreements are responsible for there being the said objects. First, observe that, just as with representational activities that concern how reality actually is now, representational activities that take place under various suppositions or that concern various times that are not now (or no time at all) might have as their (non-surrogate) subject matters facets of reality that are not (small) pluralities of objects. Second, were one to engage in such a representational activity, it would be just as beneficial to engage in it using a surrogate subject matter that consists of a (small) plurality of objects as it is to engage in a representational activity that concerns how reality actually is now using such a subject matter when its non-surrogate subject matter is something other than a (small) plurality of objects. Third, given this pair of observations, the following should be true simply in virtue of the nature of surrogacy: the fact that a representational activity takes place under some supposition or concerns some time(s) other than now (or no time at all), should not, by itself, be responsible for it being impermissible to apply or coapply a term/concept that refers to a surrogate object in that representational activity. Rather, fourth, it should be impermissible to apply or coapply a term/concept that refers to a surrogate object in a representational activity only if it is impermissible to apply or coapply a corresponding term/concept for referring to a corresponding facet of the non-surrogate subject matter of that activity in that activity. Fifth, in light of my third and fourth observations, the application and coapplication of terms/concepts that refer to surrogate objects of a particular kind should be linked to the application and coapplication of terms/concepts that refer to relevant facets of the relevant non-surrogate subject matters in a way that is in no way influenced by whether the said application or coapplication occurs as part of a representational activity that takes place under certain suppositions and/or concerns certain time(s) that are not now (or no time at all). Thus, sixth, a term/concept for referring to a surrogate object should—and thus will—possess universal coapplicability if and only if the corresponding term/concept for referring to the corresponding facet of the corresponding non-surrogate subject matter possesses universal coapplicability. For instance, given that the number of *F*s is *n* if and only if there are exactly **n** *F*s links the application and coapplication of numerals that refer to natural numbers to that of numerals that denote finite cardinalities, numerals that refer to natural numbers will possess universal coapplicability if and only if numerals that denote finite cardinalities do. Seventh, as the argument that I am about to give demonstrates, numerals that denote finite cardinalities—and, thus, numerals that refer to natural numbers—do possess universal coapplicability. To see this, observe that, in *any* representational activity, there are **zero** objects in its subject matter that are non-self-identical. Thus, the number of objects in its subject matter that are non-self-identical is zero. Accordingly, the natural number zero is in the subject matter of *any* representational activity, and, because of this, there is a plurality of objects in the subject matter of *any* representational activity that possesses cardinality **one**, i.e., the objects that are identical to the natural number zero. Thus, the natural number one is in the subject matter of *any* representational activity, and, because of this, there is a plurality of objects in the subject matter of *any* representational activity that possesses cardinality **two**, i.e., the objects that are identical to the natural number zero or the natural number one. From here it is easy to see that we can use the trick referenced above to establish the universal coapplicability of both numerals that denote finite cardinalities and numerals that refer to natural numbers. Hence, eighth, according to my institutional account, the natural numbers are necessary and atemporal existents. And, ninth, it is possible to make it the case by declaration/collective agreement that there are objects that exist necessarily and atemporally. Thus, tenth, taking Reuben's comparison between mathematical and institutional objects seriously is compatible with maintaining that mathematical objects exist necessarily and atemporally.

Another claim that I have made repeatedly is that, according to my account, mathematical objects/structures are not identical to our mental representations of them. Clearly, this is the case. According to my account, mathematical objects/structures possess the same underlying nature as the US Supreme Court and Californian corporations. We don't take the US Supreme Court to be identical to our mental representations of it, nor do we take Californian corporations to be identical to our mental representations of them. Accordingly, we shouldn't take mathematical objects/structures to be identical to our mental representations of them. As I observed earlier, these two kinds of objects possess radically different properties.

Finally, I noted earlier that most philosophers of mathematics take mathematical objects/structures to be unchanging. This, too, is true according to my institutional account of such objects/structures. The basic reason for this is that mathematical structures are surrogates for all possible systems of facets of reality that possess particular structures and the features of such systems do not change. Thus, the unchangeability of mathematical objects/structures is compatible with our mental representations of them undergoing change, just as Reuben maintains that they do.

So, in summary, I have argued that, by taking Reuben's comparison between mathematical and institutional objects seriously, we can provide an account of mathematical (and many other abstract) objects according to which humans are responsible for their existence, they aren't identical to our mental representations of them, they aren't straightforwardly physical objects, and they aren't "mysterious," "spooky," or "otherworldly" either. In fact, mathematical objects/structures are just as philosophers of mathematics have long maintained that they are unchanging abstract objects that are necessary and atemporal existents.

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AQ1. Please provide ciation for Feferman, Solomon (2014), Field, Hartry (1980), Hersh, Reuben (2014).